Three-Dimensional Instability of Elliptical Flow

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A theory is presented for Pierrehumbert's three-dimensional short-wave inviscid instability of the simple two-dimensional elliptical flow with velocity field $\mathbf{u}(x,y,z)=\Omega(-Ey,E^{-1}x,0)$. The fundamental modes, which are also exact solutions of the nonlinear equations, are plane waves whose wave vector rotates elliptically around the z axis with period $2\pi/\Omega$. The growth rates are the exponents of a matrix Floquet problem, and agree with those calculated by Pierrehumbert.

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Pierrehumbert[1] has recently discovered that a large class of two-dimensional inviscid flows with elliptical vortex cores are subject to strong three-dimensional instabilities with no short-wave cutoff. This discovery is of profound importance in the theory of turbulence, as it provides a universal mechanism whereby complex three-dimensional motion can arise directly from large-scale two-dimensional coherent structures. In the last few years, it has become clear through a number of well-documented numerical and experimental studies that such a direct transfer mechanism plays a major role in the transition to turbulence in wall-bounded shear flows[2] and free shear layers.[3–5]

The similarity of the three-dimensional breakdown processes in all the available examples led Pierrehumbert to speculate that any two-dimensional, high-Reynolds-number flow containing an elliptical vortex core would be subject to the same kind of instability. He therefore considered the stability of a two-dimensional inviscid flow with an elliptical core near the origin in which the velocity field could be approximated by

$$\mathbf{u}(x,y,z)=[(0.5+\epsilon)z,0,-(0.5-\epsilon)x],$$

where $0\leqslant\epsilon\leqslant 0.5$ is a measure of the eccentricity of the elliptical streamlines near the origin. Using a high-resolution spectral eigenvalue solver, Pierrehumbert found a family of instability modes of the form

$$\mathbf{u}'(\mathbf{x},t)=\mathbf{u}_0(\beta x,\beta z)e^{i(\beta y-\omega t)},$$

where the structure function $\mathbf{u}_0$ and growth rate $-i\omega$ are independent of the scaling parameter β as $\beta\to\infty$. The maximum growth rate is a function of the eccentricity parameter ϵ only, and increases monotonically from zero at $\epsilon=0$ (rigid rotation) to roughly 0.17 (in dimensionless units) at $\epsilon=0.4$.

The object of this Letter is to present a simple theory of Pierrehumbert's instability, clarifying its physical and mathematical nature. I shall use slightly different notation in order to simplify some of the analysis. The flow that we consider will be defined everywhere by the velocity components

$$\mathbf{u}(x,y,z)=\Omega(-Ey,E^{-1}x,0),\quad \Omega>0,\quad E>1, \tag{1}$$

which are identical to Pierrehumbert's under a change of coordinates, if we define

$$E=[(0.5+\epsilon)/(0.5-\epsilon)]^{1/2},\qquad \Omega=(0.5+\epsilon)(0.5-\epsilon). \tag{2}$$

Subject to the identification (2), the results of the two investigations are directly comparable.

When the eccentricity is $E=1$, the flow (1) becomes a state of rigid rotation about the z axis at rate Ω. As is well known in geophysical fluid dynamics,[6] such a state supports a spectrum of inertial oscillations. In a reference frame rotating with the fluid, the simplest such oscillations are plane waves whose frequency is 2Ω times the cosine of the angle θ between the wave vector and the rotation axis. Viewed from an inertial frame, the wave vector rotates about the z axis at rate Ω while the wave oscillates with intrinsic frequency $2\Omega\cos\theta$. We look for a similar type of motion in the elliptic case $E>1$.

The linearized equations for the evolution of a small inviscid perturbation $\mathbf{u}'(\mathbf{x},t)$ to the flow (1) are

$$(\partial_t+\mathbf{u}\cdot\nabla)\mathbf{u}'+\mathbf{u}'\cdot\nabla\mathbf{u}=-\nabla p',\quad \nabla\cdot\mathbf{u}'=0, \tag{3}$$

where $u_i(\mathbf{x})=A_{il}x_l$ and

$$A_{ij}=\Omega\bar{A}_{ij},\quad \bar{A}=\begin{pmatrix}0 & -E & 0\\ E^{-1} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{4}$$

We try a perturbation of the form

$$(\mathbf{u}',p')=(\mathbf{v}(t),\tilde{p}(t))\exp[i\mathbf{k}(t)\cdot\mathbf{x}] \tag{5}$$

where the time dependence of the wave vector gives rotation, etc. Then (3) becomes

$$\dot{v}_j+i\dot{k}_lx_lv_j+ik_mA_{ml}x_lv_j+A_{jl}v_l=-ik_j\tilde{p},\qquad k_jv_j=0. \tag{6}$$

The terms proportional to x_l must cancel; therefore

$$\dot{k}_l = -k_m A_{ml}, \tag{7}$$

and we are left with

$$\dot{v}_j + A_{jl} v_j = -ik_j \tilde{p}. \tag{8}$$

We can project out the pressure term by contracting (8) with the tensor $\delta_{ij} - k^{-2}k_i k_j$. Using the time derivative of the incompressibility condition

$$k_j \dot{v}_j = -\dot{k}_l v_l = k_j A_{jl} v_l, \tag{9}$$

we obtain

$$\dot{v}_i = (2k^{-2}k_i k_j - \delta_{ij}) A_{jl} v_l. \tag{10}$$

Together with Eq. (7) giving the evolution of the wave vector $\mathbf{k}(t)$, (10) is the basic equation governing the evolution of the perturbation.

It is important to notice that (10) is independent of the length of the wave vector $\mathbf{k}$. This means that any waves or instabilities that we find are completely independent of the length scale; this extends Pierrehumbert's conclusion that the structures of the instabilities are independent of length scale in the limit of large wave number. Another important point is that since we only used the time derivative of $\mathbf{k}\cdot\mathbf{v}=0$ in deriving (10), its general solutions are not necessarily incompressible, but only satisfy $\mathbf{k}(t)\cdot\mathbf{v}(t)=\text{const}$. If we seek purely exponentially growing solutions, i.e., solutions that decay exponentially as $t\rightarrow -\infty$, this constant must be zero. The physically interesting solutions of (10) are therefore automatically incompressible.

The general solution of Eq. (7) is

$$\mathbf{k}(t) = k_0(\sin\theta\cos\Omega(t-t_0), E\sin\theta\sin\Omega(t-t_0), \cos\theta), \tag{11}$$

which describes motion on an ellipse parallel to the x-y plane. The wave-vector ellipse has the same eccentricity as the streamline ellipses, but with the major and minor axes reversed. The angle θ is the minimum angle between the wave vector and the z axis, and k_0, which is irrelevant to the stability problem, is its minimum length. The delay time t_0 is an arbitrary quantity that serves only to specify the phase angle of the rotation.

With $\mathbf{k}(t)$ given by (11), Eq. (10) is a Floquet problem for $\mathbf{v}(t)$. As is well known,[7] the general solution is a linear superposition of Floquet modes of the form

$$\mathbf{v}(t) = e^{\sigma t}\mathbf{f}(\Omega(t-t_0)), \tag{12}$$

where $\mathbf{f}(\phi)$ is periodic with period 2π. The Floquet exponent σ is determined by the requirement that $e^{\sigma T}$ be an eigenvalue of the matrix $M(2\pi)$, where $M(\phi)$ is a matrix that satisfies a rescaled version of (11):

$$dM_{lm}(\phi)/d\phi = (2q^{-2}q_i q_j - \delta_{ij})\bar{A}_{jl} M_{lm}(\phi), \quad M_{ij}(0) = \delta_{ij}, \tag{13}$$

with

$$\mathbf{q}(\phi) = (\sin\theta\cos\phi, E\sin\theta\sin\phi, \cos\theta).$$

The vector $\mathbf{f}(\phi=0)$ is the eigenvector of $M(2\pi)$ corresponding to the eigenvalue $e^{\sigma T}$; for $\phi\neq 0$, $\mathbf{f}(\phi)$ is determined by

$$df_i/d\phi = (2q^{-2}q_i q_j - \delta_{ij})\bar{A}_{jl} f_l(\phi).$$

Now, the average of the trace of the matrix $(2q^{-2}q_i q_j - \delta_{ij})\bar{A}_{jl}$ over $0<\phi<2\pi$ is zero, and so the determinant of $M(2\pi)$ is unity. Also, (13) has the property that $d(q_i M_{ij})/d\phi = 0$, and hence $q_j(0) = q_i(0)M_{ij}(2\pi)$, so that one eigenvalue of $M(2\pi)$ is always unity. The two remaining eigenvalues must then be either complex conjugates with unit modulus, or real and reciprocals of each other. In fact, the evenness of $q^{-2}q_i q_j$ as a function of $\mathbf{q}$ implies that if the eigenvalues are real and unequal, they must necessarily be positive.

The instability problem thus reduces to the calculation of the matrix $M(2\pi)$ and its two nontrivial eigenvalues, which, from (13), depend only on the eccentricity E and inclination angle θ. If the nontrivial eigenvalues μ,μ' of $M(2\pi)$ are real and unequal, with $\mu > 1 > \mu'$, for given E and θ, then there exists an exponentially growing solution of (10) of the form (12), whose growth rate is given by

$$\sigma(E,\theta) = (\Omega/2\pi)\ln\mu(E,\theta). \tag{14}$$

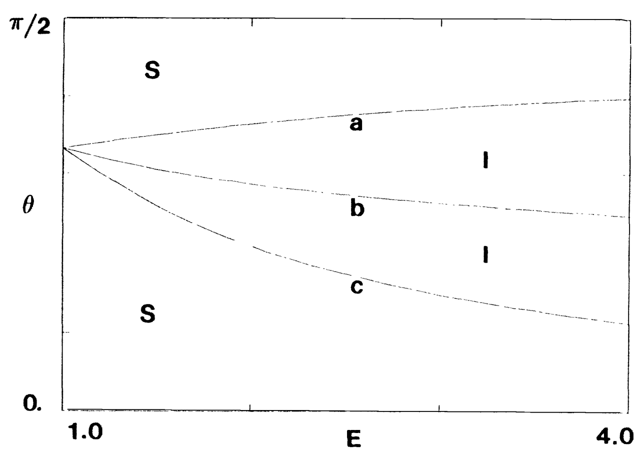

FIG. 1. Important angles for 3D instability as functions of eccentricity E. Curve a is $\theta_+(E)$, the largest angle giving instability. Curve b is $\theta_{\max}(E)$, the angle at which growth rate is maximized. Curve c is $\theta_-(E)$, the smallest angle giving instability. Labels S and I denote regions of stability and instability, respectively, in the E-θ plane.

As remarked before, such a pure exponentially growing solution automatically satisfies the incompressibility constraint $\mathbf{k}(t)\cdot\mathbf{v}(t)=0$.

In fact, $\mathbf{k}(t)\cdot\mathbf{v}(t)=0$ implies that $\mathbf{u}'\cdot\nabla\mathbf{u}'$ vanishes identically, so that the growing Floquet mode is an exact solution of the nonlinear equations. This unlimited growth is a consequence of the assumed infinite domain of elliptical flow. In the case of a finite region of approximately elliptical flow, the instability modes are linear superpositions of modes of the form (12) whose wave vectors have different delay times t_0. Such superpositions are, of course, no longer exact solutions of the nonlinear equations.

A general instability mode with wave number $k_0\cos\theta$ in the z direction may be written as

$$\mathbf{u}'(\mathbf{x},t)=e^{\sigma t}\int_0^{2\pi}d\phi_0 A(\phi_0)\exp[ik_0\mathbf{q}(\Omega t-\phi_0)\cdot\mathbf{x}]\mathbf{f}(\Omega t-\phi_0).$$

If we choose $A(\phi_0)=A=\text{const}$, then we obtain a Bessel-function-like eigenmode

$$\mathbf{u}'(\mathbf{x},t)=Ae^{\sigma t}\int_0^{2\pi}d\phi\exp[ik_0\mathbf{q}(\phi)\cdot\mathbf{x}]\mathbf{f}(\phi),$$

which decays algebraically as $k_0|\mathbf{x}|\to\infty$. By choosing k_0 large enough, we can obtain a mode localized within as small a region around the origin as desired. The eigenmodes found by Pierrehumbert[1] have an appearance that is consistent with this form.

It is an elementary numerical problem to calculate $M(2\pi)$ and its eigenvalues for any reasonable $E>1$ and $0<\theta<\pi/2$. When $E=1$, the elliptical flow reduces to rigid rotation, and we recover the familiar inertial-wave results.[6] The nontrivial eigenvalues of $M(2\pi)$ for a wave with inclination θ are $\cos(4\pi\cos\theta)\pm i\sin(4\pi\cos\theta)$, which are real and positive only when $\theta=0,\pi/3,\pi/2$. When E is slightly greater than 1, only the $\theta=\pi/3$ waves are destabilized, and there is a small band of unstable waves with θ around $\pi/3$. The band of unstable angles widens as E increases, and the growth rate at a given θ also increases with E.

Given E, we are particularly interested in the maximum value $\bar{\sigma}(E)$ of the growth rate, and the angle $\theta_{max}(E)$ at which it is attained. We are also interested in the "neutral" angles $\theta_\pm(E)$ separating the stable and unstable angles for a given value of E. In Fig. 1, I plot $\theta_+(E)$ (curve a), $\theta_{max}(E)$ (curve b), and $\theta_-(E)$ (curve c) as functions of E. As expected, the curves all originate from the point $E=1,\theta=\pi/3$. The slight downward tendency of these curves reflects the fact that θ was defined as the minimum angle between the wave vector and the z axis, while the stability properties are probably more closely associated with the average angle between the wave vector and the z axis.

In Fig. 2, I plot the growth rate maximized over θ,

$$\bar{\sigma}(E)=(1/2\pi)\ln\mu(E,\theta_{max}(E)), \tag{15}$$

as a function of E, taking $\Omega=1$. For comparison, I plot (with circles) the maximum growth rates found by Pierrehumbert. In order to compare results, I have divided his growth rates by the rotation rate $\Omega(\epsilon)$ defined by (2) and plotted it against $E=E(\epsilon)$. The results of his spectral calculation lie very close to the present curve, with a small negative bias that probably reflects the constraining nature of the boundary conditions used in the numerical formulation. The agreement between the two sets of results is surprisingly good, considering the completely different natures of the analyses, and confirms the correctness of the general conclusions.

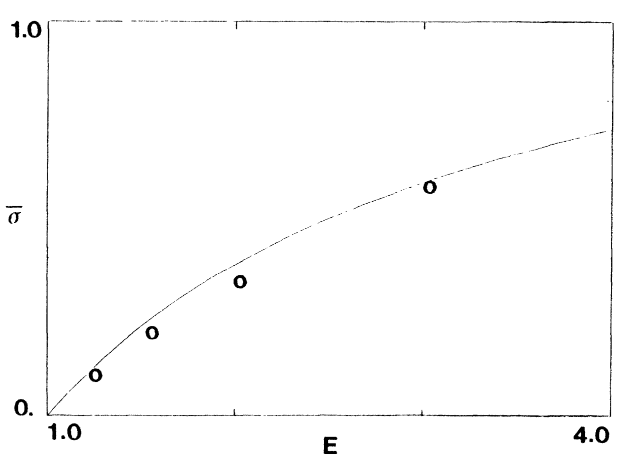

FIG. 2. Maximum instability growth rate as a function of E. The curve is $\bar{\sigma}(E)$ defined by Eq. (15), and circles are the results of Pierrehumbert's spectral calculation.

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