Universal Short-Wave Instability of Two-Dimensional Eddies in an Inviscid Fluid

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It is shown that a broad class of two-dimensional vortices occurring in the flow of an incompressible, inviscid fluid are unstable to three-dimensional perturbations. At short wavelengths along the vortex axis the growth rate becomes independent of wavelength, and the eigenmode becomes concentrated near the center of the vortex.

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The observation of organized large-scale two-dimensional eddies in a wide variety of turbulent flows[1,2] has led to a reappraisal of the fundamental nature of turbulence. Moreover, there are indications from laboratory experiments that the breakdown of these eddies catalyzes the transition to developed three-dimensional turbulence.[3] Indeed, theoretical studies have revealed that the two-dimensional structures are unstable to three-dimensional perturbations in the case of free shear layers,[4–6] isolated vortices,[7] and wall-bounded shear flow[8]; in the latter case, the instability has been implicated in subcritical transition to turbulence. These instabilities are novel in that, apart from dissipative effects, they generate arbitrarily small scales from a smooth basic state.[4,6,8] On the basis of a heuristic argument and the available examples, Orszag and Patera[8] speculated that three-dimensional instabilities are a generic feature of two-dimensional eddies. In this Letter, I exhibit a simple construction which confirms universality and accounts for the salient features of the instability.

Let the planar velocity field $\mathbf{W}(x,z) = (W_x, 0, W_z)$ be a solution of the Euler equations. I linearize the three-dimensional Euler equations about this flow; as a result of separability in y and time, one is at liberty to assume a perturbation velocity and pressure of the form $\{\mathbf{v}(x,z), p(x,z)\}\exp[i(\beta y - \omega t)]$. Let $\mathbf{V}$ be the projection of $\mathbf{v}$ on the x-z plane. The perturbation equations are

$$(-i\omega + \mathbf{W}\cdot\nabla)V + (\mathbf{V}\cdot\nabla)\mathbf{W} = -\nabla p, \quad (1a)$$

$$(-i\omega + \mathbf{W}\cdot\nabla)v_y = -i\beta p, \quad (1b)$$

together with the continuity equation $\nabla\cdot\mathbf{V} + i\beta v_y = 0$. Together with suitable boundary conditions, these define an eigenvalue problem for ω. Consider now modes with $\omega \sim 1$ as $\beta \to \infty$ and assume without loss of generality that $\mathbf{V} \sim 1$. Suppose that the modes remain smooth as $\beta \to \infty$; then the continuity equation implies $v_y \to 0$, whence (1b) implies $p \to 0$ in this limit. Since (1a) without the pressure term reduces to a family of independent differential equations along each streamline of $\mathbf{W}$, each eigenmode becomes concentrated to a streamline; this violates the smoothness assumption. Thus, in the limit $\beta \to \infty$ the eigenmodes must have an arbitrarily small-scale structure in the x-z plane. Upon introduction of the rescaled variables $x' = \beta x$, $z' = \beta z$, $p' = p\beta$, the equations become

$$(-i\omega + \beta\mathbf{W}\cdot\nabla')\mathbf{V} + (\mathbf{V}\cdot\nabla)\mathbf{W} = -\nabla' p', \quad (2a)$$

$$(-i\omega + \beta\mathbf{W}\cdot\nabla')v_y = -ip', \quad (2b)$$

together with the continuity equation $\nabla'\cdot\mathbf{V} + iv_y = 0$. ∇' is the gradient with respect to (x', z'). For modes that remain smooth along streamlines, $\beta\mathbf{W}\cdot\nabla' \sim 1$ as $\beta \to \infty$, and it is possible to have a balance at large β in which none of the terms in (2) is negligible.

Suppose that $\mathbf{W}$ has a center of rotation at (0,0), where the stream function can be expanded locally as $\Psi \approx [(\frac{1}{2} - \epsilon)x^2 + (\frac{1}{2} + \epsilon)z^2]/2$. I focus now on modes that become localized in the vicinity of $x = z = 0$ at large β; the existence of such modes will be verified numerically in due course. Near the origin $\beta\mathbf{W} = [(\frac{1}{2} + \epsilon)z', 0, -(\frac{1}{2} - \epsilon)x']$, $\nabla\mathbf{W}$ is a constant tensor, and hence β does not appear in the problem. In consequence, the eigenvalues for trapped modes become independent of β as $\beta \to \infty$, and the eigenmodes attain a self-similar form in which increasing β proportionately reduces their scale without changing their shape.

The eigenvalue problem (2) was solved numerically in the domain $|x| < 5$, $|z| < 5$ by projection on a basis of Tschebychev polynomials, truncation to a finite number of modes, and application of a standard matrix linear-algebra package; a transformed coordinate system was employed in order to increase the resolution near the origin. Zero normal-flow boundary conditions were enforced. These are mathematically consistent though artificial, but in any event have little effect on the results as the modes in question decay before the boundary is reached. The calculations were carried out on a Cyber-205 in 64-bit precision, with 20 modes in each direction. The growth rate of the most unstable trapped mode is shown as a function of ϵ in Fig. 1, and the (purely real) eigenmodes for $\epsilon = 0.1$ and $\epsilon = 0.4$ are shown in Fig. 2. Notably, the modes

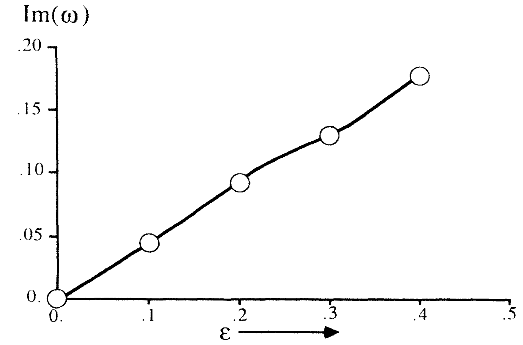

FIG. 1. Growth rate as a function of the eccentricity parameter of the vortex. Circles show values for which calculations were carried out.

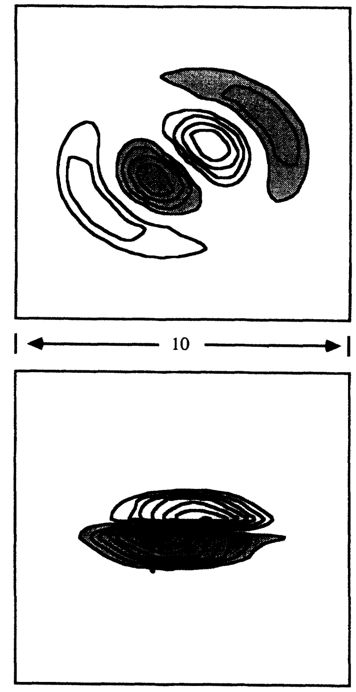

FIG. 2. Contours of perturbation y vorticity in the x'-z' plane for $\epsilon=0.1$ (top) and $\epsilon=0.4$ (bottom). Domain size is 10 units (dimensionally $10/\beta$). Shaded areas are negative.

are trapped near the origin as required. The modal structure implies bending and stretching of originally straight vortex lines into planar sinusoidal curves, and involves both the tilting and stretching required by the argument of Ref. 8. For $\epsilon=0$ the basic state consists of rigid rotation, and as expected the growth rates vanish in this limit. The growth rate increases monotonically out to $\epsilon=0.4$, and calculations with ϵ approaching 0.5 (not shown because of convergence difficulties) show no indication of the growth rates falling to zero. This is perhaps surprising, as the limit $\epsilon=0.5$ corresponds to plane Couette flow, which possesses only a spectrum of singular neutral eigenmodes and does not admit instability. The weakly nonparallel limit appears to be a singular one, and indeed an examination of the eigenmodes indicates that as $\epsilon \rightarrow 0.5$, the vorticity becomes ever more sharply concentrated on $z'=0$; this tendency is already evident in Fig. 2. It thus appears that the weak departures from parallel flow have a primarily catalytic effect, allowing the disturbance to retain a shape that can tap the kinetic energy already existing in the background shear flow.

The family of vortices considered by Pierrehumbert and Widnall[4] is parametrized by a real number ρ; the vortex with $\rho=0.25$ corresponds to $\epsilon=0.3$ and has vorticity $1/\epsilon$ at its center. For $\rho=0.25$, a growth rate of 0.39 at short waves was reported in Ref. 4. This is in reasonable agreement with the value $\omega_i/\epsilon=0.43$ obtained from Fig. 1, with consideration of the low resolution used in Ref. 4. Also, the modal structure at $\beta=2$ (see Fig. 8 of Ref. 4) is consistent with that seen in Fig. 2. Eigenfunctions for larger β were not given in Ref. 4, but subsequent calculations with the same code at $\beta=4,8$ (not shown) confirm the scaling result derived above. My results are also consistent with the findings of Orszag and Patera[8] in that (1) the instability is confined to within an $O(\beta^{-1})$ distance of the vortex center, and (2) the instability has zero phase speed in the frame moving with the vortex. Unfortunately, the information provided in Ref. 8 is insufficient to permit a meaningful comparison of growth rates.

For a constant-vorticity flow with exactly elliptical streamlines, the β independence of the growth rates and the self-similarity of the eigenmodes are exact results which do not rely on short-wave asymptotics. The extent to which these results apply to an arbitrary vortex depends on the extent to which the mode is confined to a region of the vortex within which the vorticity can be considered essentially uniform; since the trapping scale in the uniform-vorticity region decreases in inverse proportion to β, the instability as described above is expected to occur on an arbitrary smooth vortex at short waves with $\beta L >> 1$, where L is some measure of the core size of the vortex. For longer waves, the growth rate will generally depend on β. The fact that the growth rates given in Ref. 4 begin to level off when $\beta L \approx 1$ suggests that the asymptotic behavior is quite robust. The strength of the trapping in the uniform-vorticity case is difficult to determine rigorously from numerical results, but the rather slow convergence of the eigenvalues encountered as the domain size was increased to the value reported above is suggestive of algebraic rather than exponential trapping. Further, the neutral mode obtained as $\epsilon \rightarrow 0$ has the form of a pair of stationary inertial waves which superpose to have the radial dependence of a J_1 Bessel function, and thus exhibits algebraic decay.

With regard to the possible role of the instability in developed inertial-range turbulence, I remark that the short-wave instability is not incompatible with locality of energy transfer in spectral space: For fixed eccentricity, the growth rate scales with the vorticity of the eddy; hence, in a $k^{-5/3}$ energy spectrum the instability of eddies with scale k^{-1} has growth rate scaling with $k^{2/3}$. It follows that the aggregate energy transfer into a given scale due to instability of all eddies of larger scale is dominated (albeit weakly) by the nearby length scales. Seen in this light, the short-wave instability emerges as just another cascade mechanism. With regard to the *transition* to a developed turbulence spectrum, the implications are rather more novel. The short-wave behavior of the instability implies that the development of a full three-dimensional turbulence spectrum does not require that energy be handed down in a cascade from scale to scale until the dissipation range is reached; the large eddies provide a route whereby energy can be injected directly into the dissipation range.

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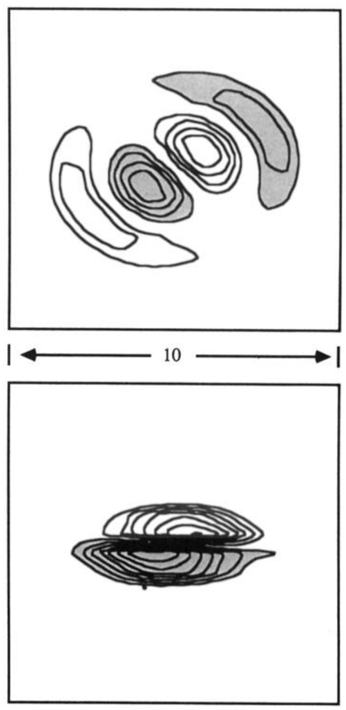

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