Evidence for Scaling in Lattice QCD at $\beta = 5.7$

J. C. Sexton and H. B. Thacker
Fermi National Accelerator Laboratory, Batavia, Illinois 60510
(Received 28 May 1986)

The SU(3) deconfinement temperature is studied on asymmetric lattices as a probe of scaling. For lattices with four sites in the temperature direction ($\beta_c \approx 5.7$) we find precise agreement between the measured asymmetry dependence of the Λ parameter and that predicted by one-loop perturbation theory. The agreement holds over a large range of asymmetry ($0.65 \leq \xi \leq 1.1$) and implies that violations of perturbative scaling above $\beta \approx 5.7$ are independent of asymmetry and therefore unlikely to be lattice artifacts. This provides evidence that the coupling range $5.7 \lesssim \beta \lesssim 6.2$ is a regime of nonperturbative but universal scaling.

PACS numbers: 12.38.Gc, 11.15.Ha

The problem of locating the onset of the scaling regime in lattice QCD is of fundamental importance since it is only in this regime that one can expect to extract continuum physics from lattice calculations. Recent Monte Carlo studies[1,2] of the deconfinement temperature in pure-gauge SU(3) have shed important new light on this issue. These studies measure the critical coupling β_c at which deconfinement occurs as a function of the number of sites, N_t, in the time or temperature direction of the lattice. The requirement that the physical deconfinement temperature $T_c = (N_t a)^{-1}$ be independent of the lattice spacing a in the scaling region means that these calculations can also be regarded as a measurement of the bare coupling as a function of that spacing. For sufficiently weak coupling, this functional dependence should be determined by the universal perturbative β function.[3] The results[1,2] strongly suggest that such perturbative scaling sets in for $\beta \gtrsim 6.2$ which corresponds to $N_t \gtrsim 12$).

For $\beta \lesssim 6.2$ things are less clear. Certainly perturbative scaling no longer holds.[4] (Indeed, the length scale determined by the deconfinement transition deviates by as much as 50% from the perturbation-theory prediction.) The question which then arises is this: Are the perturbative scaling violations observed in this regime still universal? If nonperturbative but universal scaling is in effect, then, for example, a lattice calculation of a dimensionless mass ratio at $\beta = 5.7$ will give the same value as a similar calculation at $\beta = 6.2$. Thus it would be possible to probe continuum physics at relatively small values of β, which would result in enormous savings in the computer time required to do any given lattice calculation. On the other hand, if the nonperturbative behavior observed in this regime is due to lattice artifacts then there is no reason to expect universal scaling, and realistic simulations must be performed at $\beta \gtrsim 6.2$.

The Monte Carlo calculations described in this paper are aimed at investigating perturbative scaling and its violation in more detail by the introduction of asymmetry on the lattice and the study of the deconfinement temperature as a function of that asymmetry. More specifically, we choose to work on lattices with spacing $a_x = \xi a$ along the x axis and spacing $a_y = a_z = a_t = a$ along the y, z, and t axes. In the universal scaling regime, different choices for the asymmetry ξ represent different regularization schemes, the effects of which can be summarized by an asymmetry-dependent parameter Λ. Thus if nonperturbative but universal scaling holds for $5.7 \lesssim \beta \lesssim 6.2$, then the perturbative scaling violations observed in Ref. 4 should be independent of the asymmetry after Λ has been appropriately rescaled.

The pure-gauge SU(N) lattice action for a lattice with asymmetry along the x axis is given by[5]

$$S(U) = \sum_{x,\mu<\nu} \beta \frac{a_x a_y a_z a_t}{a_\mu^2 a_\nu^2} \{1 - N^{-1} \mathrm{Re\,Tr}[U_{\mu\nu}(x)]\}, \quad (1)$$

where

$$\beta = 2N/g^2, \quad (2)$$

$$a_x = \xi a, \quad a_y = a_z = a_t = a, \quad (3)$$

and

$$U_{\mu\nu}(x) = U(x,x+\mu) U(x+\mu,x+\mu+\nu) U(x+\mu+\nu,x+\nu) U(x+\nu,x). \quad (4)$$

The matrices $U(x,x\pm\mu)$ introduced here are, as usual, $N \times N$ unitary matrices defined on the links of the lattice.

In order to relate the results that we have obtained on asymmetric lattices to those on a symmetric lattice, we first need to understand how the appropriate Λ parameters are related. The one-loop perturbative calculation of the Λ parameter on an asymmetric lattice, $\Lambda(\xi)$, relative to that on a symmetric lattice, $\Lambda(1)$, has been carried out by Karsch.[5] Following the same reasoning used in the symmetric-lattice studies of deconfinement, this result can be reinterpreted as a perturbative prediction for the gauge $\Delta\beta$ in the coupling β as the asymmetry ξ and the cutoff scale a change:

$$\Delta\beta(\xi,a)=\beta(\xi,a)-\beta(\xi=1,a_0), \quad (5)$$

where a_0 is a reference cutoff scale introduced to regulate the infrared divergences in the bare coupling.

The Karsch calculation of the Λ parameter for an asymmetric lattice employed the procedure developed by Dashen and Gross[6] for the symmetric lattice. In this procedure, the cutoff dependence of the bare coupling constant is obtained by calculating the one-loop effective action in a weak background field. Karsch presents his results in terms of three-dimensional momentum integrals. We have found it convenient to reexpress these momentum integrals as one-dimensional integrals over Bessel functions. For asymmetry along the x axis we find

$$\Delta\beta(\xi,a)=4N[c_{xy}(\xi,a)+c_{zt}(\xi,a)], \quad (6)$$

where

$$c_{\mu\nu}(\xi,a)=\int_0^\infty d\alpha[h_{\mu\nu}(\xi,a,\alpha)-h_{\mu\nu}(1,a_0,\alpha)]. \quad (7)$$

The integrands $h_{\mu\nu}$ appearing here receive the following contributions from the terms $S_{\rm sc}$, S_A, S_B, and $S_{\rm T}$ in the one-loop effective action derived in Ref. 5 (see also Ref. 6 and Hasenfratz and Hasenfratz[7]):

$$\begin{aligned}
&h_{\mu\nu}(\xi,a,\alpha)=(a_xa_ya_za_t)^{-1}(h_{\mu\nu}^{(\rm sc)}+h_{\mu\nu}^{(A)}+h_{\mu\nu}^{(B)}+h_{\mu\nu}^{(\rm T)}), \quad h_{\mu\nu}^{(\rm sc)}(\xi,a,\alpha)=-\tfrac{1}{12}N\alpha I_1^\mu I_1^\nu i_0^\sigma I_0^\tau,\\
&h_{\mu\nu}^{(A)}(\xi,a,\alpha)=+\tfrac{1}{4}N\alpha(I_0^\mu+I_1^\nu)(I_0^\nu+I_1^\nu)I_0^\sigma I_0^\tau,\\
&h_{\mu\nu}^{(B)}(\xi,a,\alpha)=+\tfrac{1}{16}N\alpha[(a_\mu^2/a_\nu^2)I_0^\mu(I_0^\nu-I_2^\nu)+(a_\nu^2/a_\mu^2)I_0^\nu(I_0^\mu-I_2^\mu)]I_0^\sigma I_0^\tau,\\
&h_{\mu\nu}^{(\rm T)}(\xi,a,\alpha)=+[(N^2-1)/8N][a_\mu^2I_0^\mu(I_0^\nu-I_1^\nu)+a_\nu^2I_0^\nu(I_0^\nu-I_1^\mu)]I_0^\sigma I_0^\tau,
\end{aligned} \quad (8)$$

where

$$I_n^\mu(\alpha)=\exp(-2\alpha/a_\mu^2)I_n(2\alpha/a_\mu^2). \quad (9)$$

The indices σ and τ introduced here refer to the two directions orthogonal to μ and ν. Note also that the subtraction at the reference cutoff scale a_0 regulates infrared divergences present in the terms $\int h_{\mu\nu}^{(\rm sc)}$ and $\int h_{\mu\nu}^{(A)}$.

The one-loop formula for $\Delta\beta(\xi,a)$ has the property that it can be written as the sum of two terms, one depending only on the cutoff scale a and one depending only on the asymmetry ξ,

$$\Delta\beta(\xi,a)=B(a)+C(\xi). \quad (10)$$

This may be seen by writing Eq. (7) as

$$c_{\mu\nu}=\int d\alpha[h_{\mu\nu}(1,a)-h_{\mu\nu}(1,a_0)]+\int d\alpha[h_{\mu\nu}(\xi,a)-h_{\mu\nu}(1,a)]. \quad (11)$$

The first integral is obviously independent of ξ. The fact that the second integral is independent of a follows from a change of variables $\alpha\to a^2\alpha$. The contribution of the first term to $\Delta\beta$ is just the familiar one-loop β-function result,

$$B(a)=2b_0\ln(a/a_0), \quad b_0=11N/48\pi^2. \quad (12)$$

We have evaluated $C(\xi)$ by numerical integration for comparison with our Monte Carlo data. Our results for the one-loop calculation are in agreement with those of Karsch.[5]

The Monte Carlo calculations for $\beta_c(\xi)$ were carried out on lattices of two and four sites in the temperature direction ($N_t=2,4$). For the case $\xi=1$ we chose the number of sites in the spatial directions to be $2N_t$. For $\xi\neq1$ the number of sites in the asymmetry direction was adjusted to keep the physical size of the lattice approximately constant ($N_x\approx 2N_t/\xi$) and thus minimize the dependence of our results on finite-size effects.

Our procedure for determining the value of β_c at each asymmetry consisted of first locating the deconfinement transition approximately by inspection of the Polyakov line variable $P(\mathbf{x})$,

$$P(\mathbf{x})=\mathrm{Tr}[U(\mathbf{x},\mathbf{x}+\hat{\mathbf{t}})\cdots U(\mathbf{x}+(N_t-1)\hat{\mathbf{t}},\mathbf{x})]. \quad (13)$$

This was followed by two or three independent Monte Carlo runs of between ten and twenty thousand sweeps each at values of β spaced by increments of 0.01 and chosen to bracket the critical value. For example, the value $\beta_c(1)=5.676\pm0.003$ for a lattice with $N_t=4$ (Fig. 2) was obtained from two independent runs at 5.67 and 5.68. The value of $|\langle P\rangle|$ was measured after each sweep and the configuration was classified as confined or deconfined. We chose $|\langle P\rangle|>0.7$ for $N_t=2$ and $|\langle P\rangle|>0.25$ for $N_t=4$ as our definition of a deconfined configuration. These values were chosen to be approximately half of the value which $|\langle P\rangle|$ attains just above β_c. [Reasonable variations of this definition result in small and fairly uniform changes in $\beta_c(\xi)$ and have a negligible effect on $\Delta\beta$.] The final result for β_c was obtained by linear interpolation to the point where exactly half of the configurations are con-

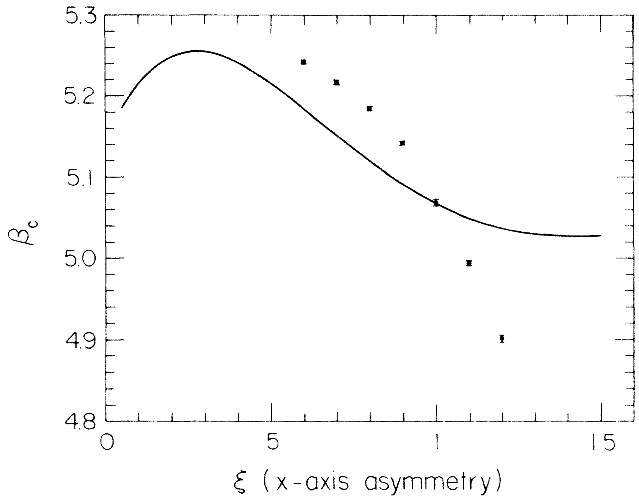

FIG. 1. The critical coupling β_c as a function of asymmetry for $N_t = 2$. The solid line represents the one-loop perturbative result. The data points represent the Monte Carlo results.

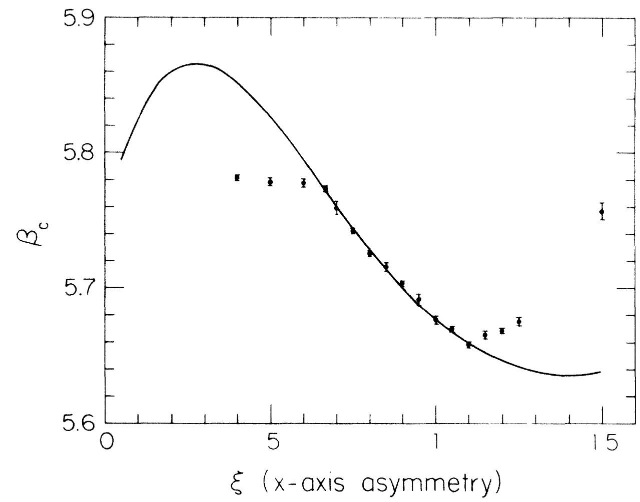

FIG. 2. The critical coupling β_c as a function of asymmetry for $N_t = 4$. The solid line represents the one-loop perturbative result. The data points represent the Monte Carlo results.

fined. The error bars that we quote on our data are purely statistical and were determined by a "jackknife" procedure[8] in which small subsets of data are eliminated and the remaining data reanalyzed. A more detailed discussion of our numerical analysis will be presented elsewhere.

The results of the Monte Carlo calculations for lattices of two sites and four sites in the temperature direction ($N_t = 2$ and 4) are plotted in Figs. 1 and 2. Also plotted in these figures are the one-loop perturbative predictions for $\beta_c(\xi)$. Note that, once the symmetric-lattice result $\beta_c(1)$ is given, the perturbative calculation for $\beta_c(\xi)$ $[= \beta_c(1) + \Delta\beta(\xi)]$ contains no adjustable parameters. For the case $N_t = 2$ (Fig. 1) the Monte Carlo results for the asymmetry dependence of β_c are in complete disagreement with the perturbative result. This is to be expected, since the values of β_c here ($\beta_c \approx 5.1$) are well into the strong-coupling region. For $N_t = 4$ (Fig. 2), the relevant values of β_c are around 5.7. From the results of Ref. 4 we know that this is still in the region where there are large violations of perturbative scaling. In view of this, the Monte Carlo results for β_c, shown in Fig. 2, are both remarkable and surprising. In the region $0.65 < \xi < 1.1$, the value of β_c follows the perturbative curve with great precision. Thus, after Λ has been rescaled to take account of the change in regularization as ξ changes we find that T_c is independent of ξ in this region. To illustrate this we have plotted the dependence of T_c on ξ in Fig. 3. Note that the region of perturbative behavior is clearly delineated at each end by sharp changes of slope which appear to be either transitions or rapid crossovers.

Before considering possible explanations for our data let us first reiterate the precise connection between our results and those of Ref. 4. In that reference, the value of β_c was measured on symmetric lattices with varying values of N_t. Since $N_t a$ is held fixed at $(T_c)^{-1}$, Ref. 4 gives the change of the bare coupling constant induced by a change of the cutoff scale, $a \to \lambda a$. In our calculations, we have held the value of N_t fixed and varied the asymmetry in one of the spatial directions. Thus, we are varying the lattice spacing in only one direction instead of all four. In terms of the quantity $\Delta\beta(\xi, a)$, our calculations study the ξ dependence while those of Kennedy *et al.*[4] study the a dependence. Thus, although the perturbative agreement that we observe is surprising, it is not inconsistent with previous results.

The striking agreement between Monte Carlo data

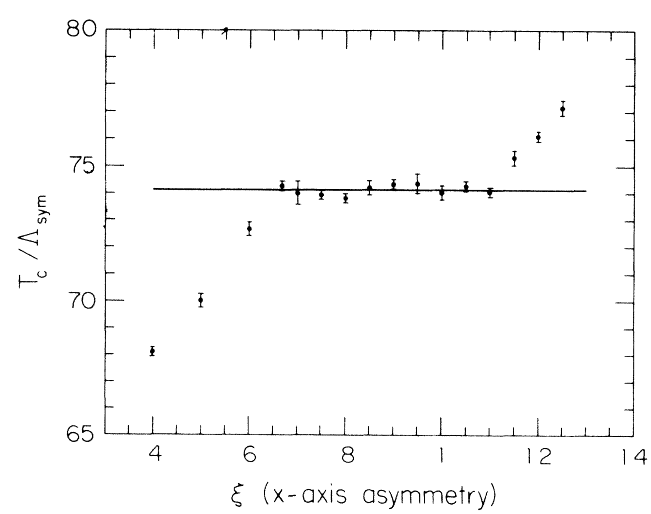

FIG. 3. T_c/Λ_{sym} as a function of asymmetry for $N_t = 4$. Λ_{sym} is the same scale as used in Fig. 3 of Ref. 1.

and perturbation theory for the ξ dependence of β_c exhibited in Fig. 2, taken together with the relatively large violations of perturbative scaling in the a dependence of β_c,[4] places strong constraints on the nature of asymptotic scaling violations in the region 5.7–6.2. One possibility is that the agreement with perturbation theory is accidental and does not indicate true perturbative behavior. (This is the case, for example, for the approximate, but accidental, equality of the values of T_c/Λ, for symmetric lattices with $N_t = 2$ and 4.) On the basis of the quality of the data and on the range of agreement with perturbation theory, we consider this possibility unlikely. We therefore expect similar agreement on lattices with six, eight, and ten sites in the time direction. (We are currently proceeding with the $N_t = 6$ Monte Carlo study.)

If we assume that the agreement with perturbation theory is not spurious, then we conclude that the full function $\Delta\beta$ has the same form as the one-loop result [Eq. (10)], i.e., it must be the sum of a term which depends only on a and a term which depends only on ξ. Moreover, in the region $5.7 \lesssim \beta \lesssim 6.2$ and for the range of asymmetries $0.65 < \xi < 1.1$, the violations of perturbative scaling are confined to the first term $B(a)$ only, while the ξ dependence is given quite precisely by the one-loop perturbative result. This may be more understandable if we observe that the separation of the one-loop result into an a-dependent term and a ξ-dependent term corresponds to a separation of the momentum-space Feynman integrals into contributions of long wavelength ($k << 1/a$) and short wavelength ($k \sim 1/a$), respectively. The term $B(a)$ in Eq. (10) is universal and ξ independent because it comes from the long-wavelength components which are not sensitive to the details of the lattice cutoff. The integrals which contribute to $B(a)$ have the form

$$\int_{-\pi/a_\mu}^{+\pi/a_\mu} d^4k\, N(k) \left[\frac{1}{(\xi a)^2} \sin^2(k_1 \xi a) + \frac{1}{a^2} \sum_{i=2}^{4} \sin^2(k_i a) \right]^{-2}, \qquad (14)$$

where $N(k)$ approaches a ξ-independent constant as $|k| \to 0$. If the region of integration is separated into $|k| < \epsilon/a$ and $|k| > \epsilon/a$ with $\epsilon << 1$, then we find that only the first of these contributes to $B(a)$. Thus our Monte Carlo result that only $B(a)$ shows nonperturbative effects strongly suggests that the observed violation of perturbative scaling in the region $5.7 \lesssim \beta \lesssim 6.2$ is not a lattice artifact, but rather an indication of nonperturbative continuum physics in this range of coupling.

We would like to thank R. K. Ellis, E. Eichten, and P. MacKenzie for helpful discussions. We are grateful to N. Hughart, S. Kalitz, D. Ritchie, and the Fermilab Computer Department, to P. Lucas, F. Nagy, L. Vanags, and the Fermilab Accelerator Division, and to G. Alverson, M. A. Cummings, D. Geesaman, J. Hanlon, C. Kaliher, W. Marsh, and N. Oshima for their assistance in utilizing the site-wide VAX network and FPS164 Array Processor on which these calculations were carried out. Some of the data were also obtained on the Magnetic Fusion Energy Computer Center Cray XMP computer at the Lawrence Livermore National Laboratory.

[1]S. A. Gottlieb, J. Kuti, D. Toussaint, A. D. Kennedy, S. Meyer, B. J. Pendleton, and R. L. Sugar, Phys. Rev. Lett. **55**, 1958 (1985).

[2]N. H. Christ and T. E. Terrano, Phys. Rev. Lett. **56**, 111 (1986).

[3]D. J. Gross and F. Wilczek, Phys. Rev. D **8**, 3633 (1973).

[4]A. D. Kennedy, J. Kuti, S. Meyer, and B. J. Pendleton, Phys. Rev. Lett. **54**, 87 (1985).

[5]F. Karsch, Nucl. Phys. **B205 [FS5]**, 285 (1982).

[6]R. Dashen and D. J. Gross, Phys. Ref. D **23**, 2340 (1981).

[7]A. Hasenfratz and P. Hasenfratz, Nucl. Phys. **B193**, 210 (1981).

[8]S. Gottlieb, P. B. MacKenzie, H. B. Thacker, and D. Weingarten, Nucl. Phys. **B263**, 704 (1986).