Is There an Upper Limit to Fermion Masses?

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A fermion mass generated by spontaneous symmetry breaking is proportional to its Yukawa coupling y to a Higgs field. Like the Higgs-field self-coupling, y may well be trivial and diverge at a *finite* energy scale Λ_f, corresponding to an upper limit on fermion masses. We verify this by solving a simple model by means of a $1/N$ expansion. Applied to the standard model, this suggests that there is an upper limit to quark and lepton masses. These results have implications for the "decoupling" of heavy fermions and bear on the issue of whether apparently "anomalous" gauge theories can be consistently quantized.

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Among the arbitrary parameters inherent in the standard model, the Yukawa couplings responsible for fermion masses seem especially artificial. Of the known particles, the couplings range from as small as 10^{-6} for the electron to 10^{-2} for the bottom quark. If the top quark exists and has a mass above 50 GeV, its associated Yukawa coupling would be greater than 0.2. One cannot help but wonder whether there might not be quarks or leptons which are extremely heavy.

There are phenomenological limits to the mass *splitting* between members of a quark or lepton doublet, due to the size of induced radiative corrections such as the vector-boson mass shifts or the ρ parameter. There are constraints on the number of generations having light neutrinos stemming from the width of the Z^0 and from certain cosmological constraints on the number of sufficiently light neutrinos. However, there may be additional generations and, if neutrinos are allowed arbitrary masses, there are no strict limits on their number. More generally, there is no upper limit on the *mean* mass of a heavy-fermion doublet without further assumptions, for example, on branching ratios or decay rates.

In this note, for reasons similar to the existence of an upper limit to the Higgs-boson mass,[1,2] we suggest that there exists an upper limit to the magnitude of fermion masses, which includes the mean mass of a multiplet. Because this involves a strong interaction, it is not a matter which can be settled in perturbation theory. Below, we substantiate this speculation in a model which can be solved via a $1/N$ expansion. However, the standard model may be amenable to strong-coupling, lattice calculations and, if our simple model is to be believed, it may be adequate to proceed in a quenched approximation, i.e., one may not need to include quantum fluctuations involving fermion loops.

In addition to the phenomenological application to the standard model, this issue bears crucially on a central theoretical question of gauge-field theory, viz., whether such theories having fermion anomalies actually can be quantized consistently and renormalizably by a suitable redefinition of the fermion measure,[3,4] a subject which we will only touch upon in this note.

Let us begin by a review of the way in which an upper limit can be realized for the Higgs-boson mass.[1,2] The Higgs-boson mass m_H is determined by the weak scale $v \approx 250$ GeV and the magnitude of the quartic coupling λ. However, there is another scale in the problem, viz., the ultraviolet cutoff Λ_c at which λ would go to infinity. To make m_H larger requires an increase in the coupling $\lambda(m_H)$ (on the scale m_H), which implies that Λ_c becomes smaller. That is, for a fixed weak scale v, the Higgs-boson mass is increased only by decreasing the cutoff Λ_c. Eventually, there comes a point beyond which the Higgs-boson mass m_H would become larger than Λ_c, at which point one must stop since it makes no sense to ascribe a mass to a particle lying beyond the cutoff on the local field theory. This crossover point is calculable in terms of the weak scale v and in fact seems to be only a few times v. It is in this sense that there is an upper limit to the Higgs-boson mass.

An analogous line of reasoning leads to an upper bound on fermion masses: In order for a fermion mass m to be very heavy, its Yukawa-coupling $y(m)$ must be large (on the scale of the fermion mass); that is, there appears a new strong interaction. However, a Yukawa-coupling y is not asymptotically free but rather tends to increase at shorter distances. In fact, as with the Higgs-boson self-coupling λ, we suspect that there is a finite scale Λ_f at which the Yukawa coupling would blow up. In principle, Λ_f could be well below the analogous scale Λ_c associated with λ. (However, because λ and y are coupled via the renormalization-group equations, the two scales are not entirely decoupled.) That is, even if the Higgs boson were very light, there could be a cutoff on the low-energy effective field theory which stems from the description of heavy quark and lepton masses via Yukawa couplings. To make m larger involves increasing $y(m)$ or, equivalently, decreasing Λ_f, as a result of which there may come a crossover point where $m > \Lambda_f$, beyond which the fermion, if it exists, cannot be described by the low-energy effective field theory. As with the cutoff Λ_c, this value is in principle determined by the weak scale v, and we will estimate it in our model below.

We illustrate these ideas with a simple model involving N left-handed and N right-handed fermions and an $N \times N$ complex Higgs field Φ_r^l. For $N=2$, this is not the minimal model but like a two-doublet model.[5] Since the gauge couplings will be assumed to be small compared to the Yukawa coupling, as a first approximation we neglect gauge interactions altogether. The Lagrangean we wish to consider is

$$\mathcal{L} = \bar\psi_L^l i\partial\!\!\!/ \psi_L^l + \bar\psi_R^r i\partial\!\!\!/ \psi_R^r + y(\bar\psi_L^l \Phi_r^l \psi_R^r + \text{H.c.}) + \mathcal{L}_{\text{H}}, \quad (1)$$

to which the addition of counterterms is tacitly implied. The Higgs-boson Lagrangean $\mathcal{L}_{\text{H}}$ need not be precisely specified but might be of the form

$$\mathcal{L}_{\text{H}} = \text{Tr}\partial_\mu \Phi^\dagger \partial^\mu \Phi - \tfrac{1}{2}\lambda[\text{Tr}(\Phi^\dagger\Phi) - v^2]^2. \quad (2)$$

For stability, we must take $\lambda > 0$ and, without loss of generality, we can assume $y > 0$ as well. There is a global $U(N) \otimes U(N)$ invariance, spontaneously broken to a diagonal $U(N)$ with y proportional to the mean mass of the fermion multiplet. To study variations with the fermion mass, we imagine keeping fixed the vacuum expectation value $\langle\Phi\rangle \equiv v1$ as we vary the Yukawa coupling h. We shall expand in powers of $1/N$. In order that the fermion mass remain of $O(1)$ in an expansion in $1/N$, we will hold y^2N, λN, and v^2/N fixed of $O(1)$. Thus it is convenient to redefine $y \to y/\sqrt{N}$, $\lambda \to \lambda/N$, $v \to v\sqrt{N}$, *so that henceforth* y, λ, *and* v *are simply independent of* N.

Our goal is to study the behavior of the fermion mass as a function of y by calculating the pole of the fermion propagator. To leading order in $1/N$, the Higgs-boson propagator and the renormalization of λ take the same forms as in Ref. 2.[6] Thus, given any nonzero coupling $\lambda(\mu)$ at an energy scale μ, there is a finite scale Λ_c at which the coupling blows up. To leading order in $1/N$, there are no corrections to the proper Yukawa vertex, i.e., the only renormalization of y is due to the fermion wave-function renormalization. Thus, $\beta_y = 2y\gamma$, where γ is the anomalous dimension of the fermion field. To leading order in $1/N$, the renormalized fermion propagator has no corrections to the mass term, and the renormalized inverse fermion propagator has the form $\mathcal{A}p\!\!\!/ - yv$. The physical mass m must be obtained by solving $\mathcal{A}^2p^2 = y^2v^2$ to determine $p^2 = m^2$. As we shall see, there is associated with y a scale Λ_f at which the coupling y blows up. For our purposes, it is sufficient to consider the case when $\Lambda_f \ll \Lambda_c$, i.e., $\lambda \ll y^2$. We may then neglect the mass of the Higgs bosons as small compared to the fermion mass. More significantly, we neglect scalar rescattering contributions compared to Yukawa interactions. (One might well worry whether fermion loops might not render the effective potential unbounded below in such a case, but they can be seen to be suppressed by at least a factor of $1/N$.) The wave-function renormalization $\mathcal{A}(-p^2)$ obeys the integral equation (see Fig. 1)

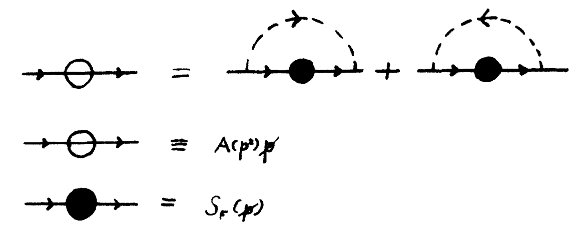

FIG. 1. Diagrams depicting the integral equation that determines the propagator function A.

$$(\mathcal{A}-1)p\!\!\!/ = y^2 \int \frac{d^4k}{(2\pi)^4} \frac{\mathcal{A}k\!\!\!/}{(\mathcal{A}^2k^2 - yv)} \frac{1}{(p-k)^2} + \text{counterterm}. \quad (3)$$

Some regularization prescription is implicitly assumed, and we have neglected the Higgs-boson mass as discussed previously. The specification of the counterterm determines, of course, the wave-function renormalization constant Z and thereby the anomalous dimension γ, but for the time being we choose to postpone a discussion of the normalization convention.

It is straightforward to convert the integral equation into the nonlinear differential equation

$$p\!\!\!/^2 \frac{d^2\mathcal{A}}{d(p\!\!\!/^2)^2} + 3\frac{d\mathcal{A}}{dp\!\!\!/^2} + \frac{\kappa^2 y^2 \mathcal{A}}{\mathcal{A}^2 p\!\!\!/^2 + y^2v^2} = 0, \quad (4)$$

where we defined $\kappa \equiv 1/4\pi$. *Here we have defined* $p\!\!\!/^2 = -p^2$ to be *positive for spacelike momenta*. This equation is a convenient starting point for determining $\mathcal{A}$ numerically since there are standard routines for solving such equations. First, however, it is useful to develop some intuition into the problem by consideration of the renormalization-group equation for $\mathcal{A}$,[7]

$$\left[\mu\frac{\partial}{\partial\mu} + \beta_y\frac{\partial}{\partial y} - 2\gamma\right]\mathcal{A} = 0. \quad (5)$$

The function $\mathcal{A}$ is of course dimensionless, so we may think of it as a function of $y(\mu)$ and dimensionless ratios, $\mathcal{A} = \mathcal{A}(p\!\!\!/^2/\mu^2, v^2/\mu^2, y(\mu))$. With use of $\beta_y = 2y\gamma$, it is easy to see that the solution has the form

$$\mathcal{A}(p\!\!\!/^2/\mu^2, v^2/\mu^2, y(\mu)) = \mathcal{A}(1, v^2/p\!\!\!/^2, Y(t;y)) \frac{y(\mu)}{Y(t;y(\mu))}, \quad (6)$$

where $t=\frac{1}{2}\ln(p^2/\mu^2)$ and Y is the running coupling constant which satisfies $\partial Y/\partial t=\beta_y(Y,t)$. For our purposes it is simplest to choose the counterterm such that $\mathcal{A}(1,\nu^2/p^2,Y(t;y))=1$, so that the momentum dependence of $\mathcal{A}$ is given entirely by an inverse-running coupling constant. With this prescription, the physical fermion mass is obtained by solving $\mathcal{A}^2 p^2=-y^2\nu^2$ or $p^2=-Y^2\nu^2$. While this normalization convention is easiest to interpret and includes the physical threshold at the fermion mass, it is more common to choose a mass-independent prescription in which β_y has no explicit dependence on t. Therefore we will discuss this situation first, and then return to the former prescription.

To understand the mass-independent case, one can simply set $\nu=0$ in the preceding equations. Choosing the counterterm so that $\mathcal{A}=1$ at $p^2=\mu^2$, one can convert the preceding second-order equation into a first-order differential equation for β_y:

$$\frac{d\beta_y(Y)}{dY}=\frac{2}{Y}\beta_y-4+\frac{4\kappa^2 Y^3}{\beta_y}. \tag{7}$$

From this, one recovers the perturbative approximation $\beta_y\approx\kappa^2 Y^3$ for small Y and finds that $\beta_y\approx\kappa Y^2(8\ln\kappa Y)^{1/2}$ for sufficiently large Y. This implies that the infrared-free Yukawa coupling does indeed blow up at some *finite* momentum Λ_f.

Now let us return to the discussion of the mass-dependent case and restore ν to the equation. This equation has been solved numerically for a wide range of coupling strengths,[8] but we can prove the existence of an upper limit to the fermion mass and give an estimate of its value rather easily. Introducing a dimensionless variable $z\equiv\kappa^2 p^2/\nu^2$ and rescaling $F\equiv\mathcal{A}/\kappa y$, Eq. (4) becomes

$$z\frac{d^2F}{dz^2}+3\frac{dF}{dz}+\frac{F}{zF^2+1}=0. \tag{8}$$

Observe that this is completely independent of the coupling constant y, so that y enters the solution only through the initial conditions for $F(z)$. This equation is obviously degenerate at $z=0$ and has both a regular and irregular solution at this point. Of course, the inverse propagator must be analytic in p^2 up to the fermion thereshold at $p^2=m^2$ (i.e., $z=-\kappa^2m^2/\mu^2<0$), which requires that we seek the regular solution of (8). We are free to specify the normalization scale, and it is convenient for our purposes to choose $\mathcal{A}=1$ at $z=0$, so that $F(0)=1/\kappa y(0)$.[6] Different possible theories correspond to different choices for the coupling strength $y(0)$. We are interested in what happens in the strong-coupling limit where $F(0)$ is small. In that case, there is a range of momenta (or z) over which $F(z)$ remains small so that the nonlinearity in Eq. (8) may be neglected. Then, one recognizes this as a transformation of the Bessel equation, and we find

$$F(z)\approx 2F(0)J_2(2z^{1/2})/z. \tag{9}$$

In particular, F vanishes at the first zero of the Bessel function J_2, viz., $z_0\approx 6.5$, *independent* of $y(0)$. Thus, for sufficiently strong coupling, the scale Λ_f where the coupling constant blow up becomes independent of the coupling strength $y(0)$![9] Clearly, for $y(0)$ sufficiently large, the linear approximation will be good throughout the spacelike momentum range from $z=0$ out to the cutoff. Now the question is, how does the fermion mass behave with increasing Yukawa coupling? The fermion mass corresponds to the solution of $F(z)^2=-1/z$. That is, in the linear approximation, we must solve

$$\frac{I_2(2(-z)^{1/2})}{(-z)^{1/2}}=\frac{\kappa y(0)}{2}. \tag{10}$$

[Even though the Bessel function I_2 grows exponentially for negative z, the linear approximation will hold out to the fermion mass for sufficiently large coupling $y(0)$.] Thus, the fermion mass m continues to grow with the increasing coupling [approximately as $2\pi\nu\ln(\kappa y)$]. This establishes that for sufficiently large coupling the fermion mass will exceed the cutoff Λ_f, so that there must be a maximum value of the fermion mass for which the fermion lies within the realm of applicability of the low-energy effective field theory. A good approximation to this maximum mass is given by the lowest value of Λ_f determined above, approximately $m\leqslant 10\pi\nu$. (This has been confirmed by more careful numerical solutions.) If one naively assumes this true for $N=2$, this corresponds to an upper limit of about 5.6 TeV.

While we believe the preceding property is likely to be generally true for masses generated by Yukawa couplings, it would be surprising if the preceding model were qualitatively correct for the standard model. The only nonperturbative method known for strong-coupling problems at fixed N is lattice simulations, and we urge that the preceding be investigated in lattice Higgs models with fermions. If the $1/N$ expansion is a reliable indication, it may be sufficient to neglect fermion loops. On the other hand, it may not be possible at finite N to ignore the Higgs-field self-coupling λ since it may have a lower limit (in perturbation theory, of order κ^2y^4 stemming from fermion loops) determined by the requirement that the effective potential be bounded from below at large fields. (Indeed, this would be another interesting matter to settle.) If one assumes that this model illustrates a property which is also true of the standard model, one may say there is an upper limit to the possible masses of quarks and leptons. Of course, whether there may exist even larger masses for them in the microscopic theory which must replace the standard model above some energy scale, one cannot say. But it does suggest that it is inherently inconsistent to assign arbitrarily large masses to fermions via spontaneous symmetry breaking.

As a result, previous discussions of fermion decoupling in such circumstances[3] must be revised and the implications for anomalies reevaluated. While this will be treated in future work, one may anticipate some of the salient features which may emerge from such an analysis. First of all, because of strong coupling, the scalars are not properly accounted for by a background field. One may not conclude that heavy fermions become infinitely heavy as one increases the Yukawa coupling, but only that they exceed a finite energy cutoff Λ_f whose value is proportional to the scale of symmetry breaking v. Second, the apparently anomalous effective field theory including the Wess-Zumino terms applies only below this cutoff. In this view the Wess-Zumino terms, like other nonrenormalizable interactions, are indicative of an energy scale at which the description in terms of low-energy excitations breaks down and new degrees of freedom enter. In the present context, either one reaches a scale m at which the heavy fermions are present or else the description in terms of these elementary fields becomes invalid (beyond Λ_f). (Speculations about the existence of heavy solitons[3] may also be beyond the scale of applicability of the effective Lagrangean.)

One cannot deduce the form of the underlying microscopic theory, but it seems plausible from these considerations that, starting from a gauge-field theory which is not anomalous, one cannot generate an apparently anomalous theory which is valid on all scales. Fermion "decoupling" therefore could not be used as evidence that an anomalous gauge theory can be consistently quantized, a subject about which there has been considerable speculation recently.[4] Of course, Faddeev and Shatashvili's ideas apply to a theory comprised only of non-Abelian gauge fields and chiral fermions, and does not require the inclusion of scalar fields that lead to spontaneous symmetry breakdown. The importance of our observations is the suggestion that one cannot arrive at such a theory starting from spontaneous breakdown of a Higgs-like sector by pushing some of the fermions off to infinity. The issue then is whether the situation without a Higgs field is truly different from the one considered here and in Ref. 3. At first sight, they appear very different, but in fact the conjectured resolution of the Faddeev problem involves the addition of an "auxiliary" scalar field and a Wess-Zumino type of term. So the main distinction seems to be whether this scalar, Goldstone-type field appears with kinetic energy and possibly other terms. In two dimensions, these have been shown to be present in the anomalous Schwinger model[8] and they could well arise generally.[10]

Thus the two situations are not really so distinct as they superficially appear, and we conjecture that they are in fact, equivalent. *If so, an apparently anomalous gauge theory would always be a nonrenormalizable, low-energy effective field theory approximation to a renormalizable, underlying local field theory that involves additional fermions which cancel the anomaly.* This conclusion may be reassuring to superstring theorists as well, for whom anomaly cancellation is a vital selection criterion. This work was supported in part by the U. S. Department of Energy. A portion of this work was carried out while visiting the Hebrew University of Jerusalem under the auspices of the Racah Institute of Physics and the U.S.-Israel Binational Science Foundation. We would like to thank Professor D. Amit and Professor E. Rabinovici for their warm hospitality.

[1]From the lattice approach, this was first suggested by R. Dashen and H. Neuberger, Phys. Rev. Lett. **50**, 1897 (1983). A review of more recent lattice Higgs models is given by J. Jersak, review talk given at the Workshop on Lattice Gauge Theory, Wuppertal, West Germany, 1986, Technische Hochschule, Aachen Report No. 85/25 (to be published).

[2]From the point of view of the $1/N$ expansion, this was suggested by M. B. Einhorn, Nucl. Phys. **B246**, 75 (1984). See also the seminar in Theoretical Advanced Studies Institute Lectures in Elementary Particle Physics, Ann Arbor, Michigan, 1984, edited by D. N. Williams (unpublished).

[3]This possibility was first raised in the context of decoupling of a fermion whose mass is generated by a Yukawa coupling by E. D'Hoker and E. Farhi, Nucl. Phys. **B248**, 59, 77 (1984).

[4]This idea has been reconsidered from the point of view of consistently quantizing a gauge theory with the Gauss's law constraint. See L. D. Faddeev and S. L. Shatashvili, Phys. Lett. **167B**, 225 (1986), which contains references to related literature.

[5]This is simpler to analyze for our purposes and, since we will assume that the Higgs bosons are light, we think that it makes little difference dynamically.

[6]Unlike the mass-independent prescription, now $y(0)$ is nonvanishing.

[7]There are scalar mass μ and self-coupling λ renormalizations but no scalar wave-function renormalization to this order. However, the ratio $v=\mu/\sqrt{\lambda}$ is scale independent so that $\beta_v=0$.

[8]I. G. Halliday, E. Rabinovici, A. Schwimmer, and M. Chanowitz, Nucl. Phys. **B268**, 413 (1986). See also M. Chanowitz, Phys. Lett. **B171**, 280 (1986).

[9]This conclusion is peculiar to this mass-dependent prescription and is not true for the mass-independent coupling constant discussed previously.

[10]See K. Harada and I. Tsutsui, Tokyo Institute of Technology Report No. TIT/HEP-94, 1986 (to be published).