Equivalence of Active and Passive Gravitational Mass Using the Moon

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A limit is established for the violation of the equality of passive and active gravitational mass. Our test is based on an asymmetry in the composition of the moon. We hypothesize that the 2-km offset between the moon's center of figure and center of mass indicates an asymmetry in the distribution of Fe and Al. Unless the Fe on one side and the Al on the other attract each other with the same force, the moon will not be in the orbit predicted by classical mechanics. Using the results from laser ranging and a model for the moon's interior we find that the ratios of active to passive mass for Fe and Al are the same to a precision of 4×10^{-12}.

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In Newton's law of gravitation, $F_m = GmM/r^2$, it is convenient to distinguish the source of the field from the object upon which the field acts. Thus in considering the Earth's orbit about the sun, the active gravitational mass M is of the sun and the passive gravitational mass m is of the Earth. A fundamental axiom of Newtonian theory as well as of general relativity is that, for a given body, the ratio $M/m = 1$ regardless of the body's composition. In a system composed of two bodies A and B of differing composition,

$$S(A,B) = M_A/m_A - M_B/m_B = 0. \qquad (1)$$

Some time ago Kreuzer[1] tested this equality for bromine and fluorine and concluded that $S(\mathrm{F},\mathrm{Br}) < 5\times10^{-5}$. Recently Luther has achieved comparable accuracy by the direct substitution of alumina balls for tungsten ones as the active masses in a precise Cavendish balance.[2]

These laboratory limits on S make the Kruezer effect unique among the measurements that determine the parameters of a post-Newtonian theory of gravitation. As Will has noted, the equality of passive and active gravitational mass is the only input to the parametrized post-Newtonian formalism that is determined by laboratory rather than astronomical phenomena.[3] While searching for ways to improve upon a laboratory measurement of S, we have found an astronomical test which improves the existing limit by a factor of about 10^6.

Our test is based upon Bondi's observation that a breakdown of the equivalence of passive and active gravitational mass requires a violation of Newton's third law as well.[4] Imagine two isolated massive spheres of different materials A and B separated by a distance r. If $S(A,B)$ does not equal 0, the force F_{AB} which B exerts on A will not be equal to the force F_{BA} which A exerts on B. Consequently there will be a net self-force on the center of mass of the system, $F_s = S(A,B)Gm_A m_B/r^2$. The system will then accelerate in response to this self-force.

The system in our test is the moon. The dark, iron-rich basaltic maria which mark the side facing the Earth are absent from the far surface, which is almost entirely aluminum-rich anorthositic highlands. This superficial asymmetry indicates a fundamental asymmetry. The center of mass of the moon is displaced from the center of figure. According to Bills and Ferarri, this displacement is 1.98 ± 0.06 km in a direction $14° \pm 1°$ to the east of the vector pointing to the Earth.[5] The simplest model which is consistent with this offset is the eccentric core model of Wood[6] and Kaula *et al.*[7] This model assumes a two-component moon: a spherical mantle of density 3.35 g/cm^3 whose center is offset about 10 km from the center of figure determined by a crust of density 2.9 g/cm^3 (see Fig. 1).

At first we make the simplifying assumption that the mantle has the same composition as the maria and the crust has the composition of the highlands. A straightforward calculation is then made of the self-force on the moon F_s for an assumed value of $S(\mathrm{Fe,Al})$. Since F_s is tightly limited by measurements of the moon's orbit as determined by laser-ranging techniques, we use the limit on F_s to infer a limit on $S(\mathrm{Fe,Al})$. Finally, we discuss how our limit is affected by a more realistic, multicomponent moon.

To calculate the self-force on the two-component moon let the radii, densities, masses, and volumes of the crust and mantle be a and b, ρ_a and ρ_b, M_a and M_b, and V_a and B_b, respectively. Then $M = M_a + M_b$ is the mass of the moon and $V = V_a + V_b$ is its volume. Further let the origin of the local coordinate system be at the center of figure of the moon, the moon's center of mass be at $z = s$, and the center of the mantle be at $z = t$. The force on the mantle due to the crust is

$$\mathbf{F}_b = \int \rho_b \mathbf{f}_b \, dV_b, \qquad (2)$$

where $\mathbf{f}_b$ is the force per unit mass.

In calculating f_b it is convenient to imagine that the mantle is superimposed upon an homogenous sphere of radius a and density ρ_a. This corresponds to an imaginary extension of the crust into the mantle. Gauss's law then gives $\mathbf{f}_b = -(4\pi/3)G\rho_a z\hat{\mathbf{k}}$. Such an f_b is appropriate for this problem because the net force F_b' arising from this fictitious extension is just that between two concentric spheres of radius b and is zero by symmetry. Using this f_b in Eq. (2), we find $\mathbf{F}_b - (4\pi/3)G\rho_a\rho_b tV_b\hat{\mathbf{k}}$.

If the Kreuzer coefficient $S(a,b)=0$, there is an equal and opposite force which the mantle exerts on the crust and consequently the self-force on the moon is $\mathbf{F}_s = \mathbf{F}_a + \mathbf{F}_b = 0$. Alternatively, if $S(a,b)$ does not equal 0, $\mathbf{F}_a = -\mathbf{F}_b + S(a,b)\mathbf{F}_b$ and

$$\mathbf{F}_s = S(a,b)\mathbf{F}_b = -(4\pi/3)G\rho_a\rho_b tV_b S(a,b)\hat{\mathbf{k}}. \quad (3)$$

Fortunately the unknown product tV_b in Eq. (2) can be found from the location of the c.m.,

$$M_s = (\rho_b - \rho_a)tV_b. \quad (4)$$

Thus

$$\mathbf{F}_s = (4\pi/3)S(a,b)G\rho_a\rho_b Ms(\rho_b-\rho_a)^{-1}\hat{\mathbf{k}}. \quad (5)$$

Compare F_s to the force $F_M = GM_E M/r^2$ which the Earth exerts on the moon:

$$\frac{F_s}{F_M} \approx S(a,b)\frac{M}{M_E}\frac{r^2}{a^2}\frac{s}{a}\frac{\rho}{\Delta\rho} = 5S(a,b).$$

Here we have assumed that $\Delta\rho = \rho_b - \rho_a << \rho_b$, $M/M_E = \frac{1}{80}$, $r/a = 220$, $s/a = 0.0011$, and $\rho/\Delta\rho = 7$.

The main influence of F_s on the moon's orbit comes not from its radial component F_r, but from its tangential component, $F_t = F_s\sin(14°)$ (see Fig. 1). In distinction to f_r, F_t will cause a continuous increase in the moon's orbital angular velocity ω about the Earth. To see this, consider the change in total energy of the moon per lunar sidereal month:

$$2\pi rF_t = \Delta E = \tfrac{1}{2}\Delta V = (2r^2)^{-1}GM_E M\,\Delta r = \tfrac{1}{2}F_M\,\Delta r.$$

Thus the monthly fractional change in the distance from Earth to moon is $\Delta r/r = 4\pi F_t/F_M$. It is convenient to use Kepler's law $\omega^2 r^3 = \text{const}$ to express this relation as $\Delta\omega/\omega = 6\pi F_t/F_M$.

Largely because of tides on the Earth, the value of $\dot{\omega}$ observed from lunar laser ranging[8] is -25.3 ± 1.2 arc sec/century^2. LAGEOS satellite laser-ranging data recently has allowed Christodoulidis *et al.*[9] to determine the effect of ocean tides on the moon's orbit. They find a tidal effect of $\dot{\omega} = -25.5$ arc sec/century^2 before allowing for unmodeled sideband effects and dissipation within the moon.[9] Since these effects may make the agreement worse by about 1 arc sec/century^2 and since the corresponding uncertainty in the difference is roughly 2 arc sec/century^2, we have $\Delta\omega/\omega$

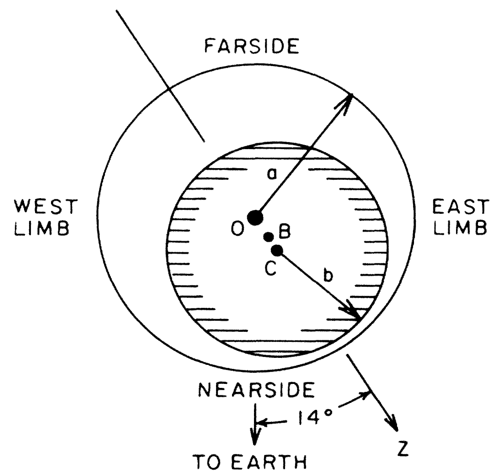

FIG. 1. Two-component model of the moon. Mantle (shaded) is eccentric with respect to crust (unshaded). O = center of figure; B = center of mass; C = center of mantle; OB = s; OC = t.

$< 1\times 10^{-12}$/month. Thus

$$S(a,b) < \tfrac{1}{5}[1/\sin(14°)](1/6\pi)(1\times 10^{-12}) = 5\times 10^{-14}.$$

We use this limit to infer S(Al,Fe). Assume that the composition of the crust (a) is the same as that of the highlands and that of the mantle (b) is the same as the maria. The most significant observed difference between the highlands and the maria is in abundances of iron and aluminum. The fractions, by weight, of Al (Fe) in the highlands and maria are 14.2% (3.4%) and 5.8% (10.8%), respectively.[10] The difference in the fractional content of Al and Fe thus changes by about 0.08 in going from the highlands to the maria. Thus

$$S(\text{Al},\text{Fe}) < 5\times 10^{-14}/0.08 = 7\times 10^{-13}. \quad (6)$$

This result was obtained from a simple two-component moon. Although we cannot obtain a model-independent result, we can greatly relax the simple two-component model. Consider the crust (a) and the mantle (b) to be eccentric as before. Now, however, allow the mantle to be divided like an onion skin into N concentric layers of radii b_i, $i = 1, \ldots, N$. Such a moon could evolve if the young, symmetric moon were ablated asymmetrically by meteorites.[6] Further assume that there is a one-to-one correspondence between the density of the ith layer and its composition. Although ridiculous for the Earth, this assumption is reasonable for the smaller moon which has insufficient temperature and pressure variations to change the density of a given mineral by more than 1–2%.[11]

The mutual interactions of the various concentric layers of the mantle will by symmetry contribute nothing to the self-force F_s. Each layer will, however, interact with the crust to produce a self-force given by Eq. (3). The total self-force is then

$$\mathbf{F}_s = -(4\pi/3) G \rho_a t \sum_i \rho_{bi} S(a,b_i) V_{bi} \hat{\mathbf{k}}. \quad (7)$$

By contrast the location of the c.m. of the moon is given by

$$Ms = t \sum_i (\rho_{bi} - \rho_a) V_{bi}. \quad (8)$$

In general Eq. (8) is no help in simplifying Eq. (7). But if

$$S(a,b_i) = K(\rho_{bi} - \rho_a)/\rho_{bi}, \quad (9)$$

where K is a constant, then $F_s = (4\pi/3) G \rho_a K M s$.

We have investigated whether Eq. (9) is in fact satisfied for common minerals on the moon and comon conjectures for S. For the latter we assume that if the strong or electromagnetic interactions lead to a nonzero S they will do so through terms proportional to either the binding energy per nucleon or the neutron-to-proton ratio (N/Z). For weak interactions,[12] we assume an effect proportional to NZ/A^2.

In Table I we list the common minerals of the moon together with their composition. Actual rocks from the moon are observed to be linear combinations of these minerals. Finally, in Fig. 2 we present three graphs showing how well Eq. (9) is obeyed for each of the three hypotheses about S. Figure 2 shows a fortuitous agreement between the curves representing Eq. (9) and the minerals on the moon. In particular, the feldspars and pyroxenes are known to be dominant on the moon's crust. These must be balanced by heavier materials in the mantle in order to give the observed average density for the moon. The current candidates are the olivines in the upper mantle and iron and troilite in the lower.[13] The relative abundances of these are immaterial since they all lie fairly close to the curves of Eq. (9). The closeness of all the minerals to these curves makes us believe that our earlier limit on S(Al,Fe) is substantially correct, even for a multicomponent moon.

Our formal limit of 7×10^{-13} on S(Al,Fe) must be relaxed because of limits in our knowledge of both the

TABLE I. Symbol, name, chemical formula, and density (g/cm^3) for common lunar materials. The bottom section gives the elemental abundances (in parts per 10^3, by weight) of the moon crust and mare.

	Feldspars	
Or Orthoclase	$AlKSi_3O_8$	2.56
Al Albite	$NaAlSi_3O_8$	2.62
An Anorthite	$CaAl_2Si_2O_8$	2.76
	Pyroxenes	
En Enstatite	$Mg_2Si_2O_6$	3.18
Di Diopside	$CaMgSi_2O_6$	3.24
He Hedenbergite	$CaFeSi_2O_6$	3.55
Fs Ferrosilite	$Fe_2Si_2O_6$	3.95
	Oxides	
Il Ilmenite	$FeTiO_3$	4.79
	Olivines	
Fo Forsterlite	$Mg_2Si_2O_4$	3.22
Fa Fayalite	$Fe_2Si_2O_4$	4.39
	Inner mantle	
Tr Troilite	FeS	4.74
Fe Iron	Fe	7.86
	Moon	
Crust	Si-217 Ti-001 Al-142 Fe-034 Mg-038 Ca-104 Na-009 O-454	2.90
Mare	Si-183 Ti-058 Al-058 Fe-108 Mg-069 Ca-099 Na-002 O-417	3.35

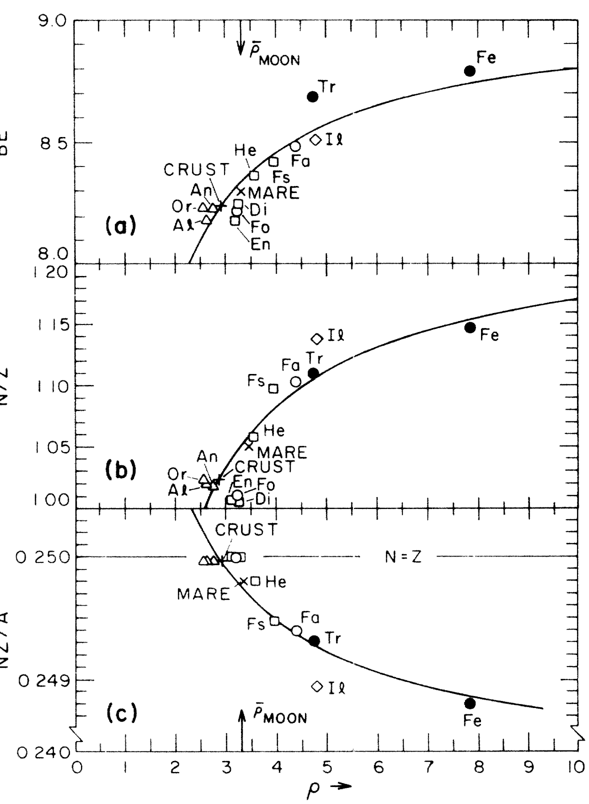

FIG. 2. Nuclear parameters (averaged by weight) for common minerals of the moon. (a) Binding energy, (b) N/Z, and (c) NZ/A^2 vs ρ. The curves are best fits with Eq. (9). The data for Eq. (10) are not shown, but closely resemble (a).

interior and the surface of the moon. Unexpected materials within the moon or first-order variations $Y_{11}(\theta,\phi)$ in the lateral positions of known minerals will compromise our onion-skin model.[14] Detailed photogrammetric mapping of the far side of the moon may modify the 14° angle of the c.m.–center-of-figure offset. Currently the east-west component of the offset is established from complete lunar orbiter altimeter measurements in the equatorial region, but photogrammetric measurements at high latitudes on the far side are confined to only the eastern part.[5] Consequently, we set a realistic limit of $S(\mathrm{Al,Fe}) = 4\times10^{-12}$.

In comparing our result with that of Kreuzer and of Luther one should realize that the laboratory materials which they used are more different in their nuclear properties than the Al-Fe comparison allowed by the moon. Specifically, Will[15] has shown that the parametrized post-Newtonian parameter ζ_3 is related to the ratio of active to passive mass by the equation

$$M/m = 1 + (3.8\times10^{-4})\zeta_3 Z(Z-1)A^{-4/3}. \qquad (10)$$

When applied to Kreuzer's flourine-bromine comparison and to our results this formula yields $|\zeta_3| < 6\times10^{-2}$ and $|\zeta_3| < 1\times10^{-8}$, respectively.

Lastly, throughout the above we have assumed that $\dot{G} = 0$. Experimental measures of $\dot{G}$ using Viking-lander data to measure the distance from the Earth to Mars have shown that $\dot{G}/G < 10^{-11}$/yr or 10^{-12}/month.[16,17] Such a value of $\dot{G}/G$ would cause a similar acceleration of the moon, $\dot{\omega}/\omega$. This value is comparable with the existing limit on an anomalous $\dot{\omega}/\omega$. Thus if a nonzero value for an anomalous $\dot{\omega}/\omega$ is ever established, an independent experiment will be needed to assign the anomaly to a violation of the equivalence of passive and active mass or to a nonzero $\dot{G}$.

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[14]Note, however, that it is the *second-order* harmonics $Y_{22}(\theta,\phi)$, corresponding to an ellipsoidal surface (or a quadrupolar distribution of mass), that are responsible for the fact that the moon always has the same face towards the Earth (see Ref. 5).

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