# Self-Consistent Dynamolike Activity in Turbulent Plasmas

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(Received 24 June 1985)

The evolution of turbulent plasmas is investigated within the framework of resistive magnetohydrodynamics. The functional form of the mean electric field is derived for fluctuations generated by tearing and resistive-interchange modes. It is shown that a bath of such local and global modes in pinches causes toroidal field reversal with finite pressure gradients in the plasma.

PACS numbers: 51.10.+y, 51.60.+a, 52.25.Gj, 52.55.Ez

It is widely believed that sustainment of a mean magnetic field in a turbulent plasma in spite of the decay caused by resistive diffusion is due to the generation of a mean electric field by the plasma fluctuations. The phenomenon, known as the dynamo effect,[1] has been invoked to explain the observed lifetimes of magnetic fields in astrophysical and laboratory plasmas, a particularly interesting example of which is the reversed-field pinch (RFP).[2] In the RFP, the plasma is observed to relax into a state in which the toroidal field is reversed near the edge, and remains in that state for times longer than can be accounted for by classical resistive diffusion alone. Evidence of such a phenomenon has also appeared in numerical simulations.[3,4]

In this Letter, we give a self-consistent and quantitative description of the mean electric field $\mathbf{F}$ generated by resistive magnetohydrodynamic (MHD) turbulence. We consider a wide class of resistive modes, global as well as those localized on field lines, and show that $\mathbf{F}$ must satisfy two important global properties which restrict the functional form of $\mathbf{F}_\parallel$, the component parallel to the mean magnetic field $\mathbf{B}_0$. This form is[5]

$$\mathbf{F}_\parallel = B_0^{-2}\mathbf{B}_0 \nabla\cdot(\kappa^2\nabla\lambda), \tag{1}$$

where $\lambda = \mathbf{J}_0\cdot\mathbf{B}_0/B_0^2$ ($\mathbf{J}_0 = \nabla\times\mathbf{B}_0$ is the mean current density) and $\kappa^2$ is a spatially varying, positive function. From a model of linear tearing modes, we derive an explicit expression for $\kappa^2$. We emphasize, however, that the realm of validity of the functional form (1) is considerably greater than that of the model calculation. Rather remarkably, it encompasses the calculations of $\mathbf{F}_\parallel$ in the work of Jacobson and Moses[6] and Strauss,[7] which differ from the model presented here and from one another, and yet obtain similar results up to different realizations for $\kappa^2$.

From a model of nonlinear, resistive-interchange modes,[8,9] we give an exact expression for $\mathbf{F}_\perp$ by extending the similarity method developed in Ref. 9. (We believe that this is the first exact, nonlinear, self-consistent calculation for $\mathbf{F}_\perp$ in the literature.) Combining $\mathbf{F}_\parallel$ and $\mathbf{F}_\perp$, we show that steady-state solutions in an RFP plasma driven by an external electric field show toroidal field reversal, generally in the presence of finite pressure gradients. In particular, the Woltjer-Taylor[10,11] force-free state $\mathbf{J} = \lambda_0\mathbf{B}$ is obtained as a special limit of our theory.

To fix ideas, we consider an incompressible plasma inside a perfectly conducting cylindrical shell (aligned with the $z$ axis), to which the velocity and magnetic fields are tangential. All mean quantities depend only on the radius $r$, and the mean magnetic field lies on surfaces $r=\text{const}$. (A generalization of the following results to compressible plasmas will appear elsewhere.) Superimposed on the mean fields are small fluctuations of zero mean, denoted by subscript 1, and varying on a faster time scale and a shorter length scale. We will use the following assumptions: (1) The mean state, containing only small flows induced by diffusion, evolves on a resistive time scale and varies spatially on the scale of the minor radius $a$ of the cylinder, i.e., $a|\mathbf{v}_0| = O(\eta)$, $a^2\partial/\partial t = O(\eta)$, $a|\nabla| = O(1)$; (2) the energy of the fluctuations, be it magnetic, kinetic, or thermal, is smaller than the mean magnetic energy, i.e., $|\mathbf{B}_1|^2, \rho|\mathbf{v}_1|^2, |p_1| << |\mathbf{B}_0|^2$; and (3) the fluctuations vary on a smaller space scale and a faster time scale than the mean quantities, i.e., $a^2\partial/\partial t >> O(\eta)$, $a|\nabla| >> O(1)$. We note that since $\mathbf{J}_1 = \nabla\times\mathbf{B}_1$, the fluctuating current may be large; we allow $a|\mathbf{J}_1|/|\mathbf{B}_0| = O(1)$. Averaging Ohm's law, $\partial\mathbf{B}/\partial t + \nabla\times(\eta\mathbf{J} - \mathbf{v}\times\mathbf{B}) = 0$, where $\mathbf{v}$, $\mathbf{B}$, and $\mathbf{J}$ are respectively the total velocity, magnetic field, and current density, we get

$$\partial\mathbf{B}_0/\partial t + \nabla\times(\eta\mathbf{J}_0 - \mathbf{F}) = 0, \quad \mathbf{F} = \langle\mathbf{v}_1\times\mathbf{B}_1\rangle. \tag{2}$$

The average may be viewed either as average over an ensemble, or as a space-time average over the scales of fluctuations. We note that $\mathbf{v}_0$ does not enter Eq. (2) because of incompressibility. We now state the two theorems concerning $\mathbf{F}$ as follows: Theorem I,

$$\int d\tau\,\mathbf{F}\cdot\mathbf{J}_0 = -\int d\tau\,\eta\langle J_1^2\rangle, \text{ to } O(\eta).$$

Theorem II,

$$\int d\tau\,\mathbf{F}\cdot\mathbf{B}_0 = 0, \text{ to } O(\eta).$$

The integrals are taken over the volume of the plasma. We point out that though these theorems have been *assumed* to hold elsewhere,[12] ours is the first rigorous demonstration, from dynamical considerations, of their validity.

Theorem I may be proved by averaging at first the energy equation

$$\partial(\tfrac{1}{2}B^2+\tfrac{1}{2}\rho v^2)/\partial t+\nabla\cdot[\tfrac{1}{2}\rho v^2\mathbf{v}+p\mathbf{v}+(\eta\mathbf{J}-\mathbf{v}\times\mathbf{B})\times\mathbf{B}]+\eta J^2=0 \tag{3}$$

($\rho$ is the density and $p$ is the plasma pressure), and keeping terms to $O(\eta)$. We get

$$\partial\tfrac{1}{2}B_0^2/\partial t=-\eta J_0^2-\eta\langle J_1^2\rangle+\nabla\cdot\mathbf{c}_1, \tag{4}$$

where $\mathbf{c}_1$ represents an energy flux. Now, from Eq. (2), we get

$$\partial\tfrac{1}{2}B_0^2/\partial t=\mathbf{F}\cdot\mathbf{J}_0-\eta J_0^2+\nabla\cdot\mathbf{c}_2. \tag{5}$$

The surface integral $\int(\mathbf{c}_1-\mathbf{c}_2)\cdot\hat{\mathbf{n}}\,dS$ represents the energy flux due to fluctuations which, for a perfectly conducting wall, vanishes. From (4) and (5), theorem I follows.

Theorem II may be proved similarly by an averaging of the exact helicity equation (with $\mathbf{B}=\nabla\times\mathbf{A}$)

$$\partial(\mathbf{A}\cdot\mathbf{B})/\partial t+2\eta\mathbf{J}\cdot\mathbf{B}+\nabla\cdot\mathbf{c}_3=0, \tag{6}$$

where $\mathbf{c}_3$ represents a helicity flux. From Eq. (2) and the corresponding dynamical equation for $\mathbf{A}_0$, we get

$$\partial(\mathbf{A}_0\cdot\mathbf{B}_0)/\partial t+2(\eta\mathbf{J}_0-\mathbf{F})\cdot\mathbf{B}_0+\nabla\cdot\mathbf{c}_4=0. \tag{7}$$

$\int(\mathbf{c}_3-\mathbf{c}_4)\cdot\hat{\mathbf{n}}\,dS$ represents the helicity flux due to fluctuations which again vanishes. From (6) and (7), we obtain theorem II.

Theorem I implies that F does *negative* work on the plasma, and is thus a dissipative force. It accounts for the Ohmic losses due to the fluctuations. We now give a *heuristic* derivation of the functional form of $\mathbf{F}_\parallel$ given by Eq. (1), also derived independently in Ref. 12. From theorem II, we may write $\mathbf{F}_\parallel=|\mathbf{B}_0|^{-2}\mathbf{B}_0\nabla\cdot\boldsymbol{\zeta}$, where $\boldsymbol{\zeta}\cdot\hat{\mathbf{r}}=0$ on the plasma boundary. (We note that, in addition, $\nabla\cdot\boldsymbol{\zeta}$ itself vanishes, as $\mathbf{F}_\parallel$ must, on the perfectly conducting wall. This condition will be shown to be immaterial when we deal with the numerical calculation given later in the paper.) For a nearly force-free plasma, theorem I now implies $\int\boldsymbol{\zeta}\cdot\nabla\lambda\,d\tau>0$, $\lambda=\mathbf{J}_0\cdot\mathbf{B}_0/B_0^2$. We envision the turbulence as generated by a bath of resistive modes, where it is known that each mode behaves nonideally only in the vicinity of some resonant magnetic surface. The contribution to mean quantities such as $\boldsymbol{\zeta}$ comes only from the vicinity of the surface. Therefore $\int\boldsymbol{\zeta}\cdot\nabla\lambda>0$ should be satisfied by the integrand itself being positive, which leads us to expect that $\boldsymbol{\zeta}=\kappa^2\nabla\lambda$, where $\kappa^2$ is a positive function. Thus, Eq. (1) follows. (Special realizations for $\kappa^2$ appear in Ref. 6, in which a kinetic model is used, and in Ref. 7, based on a quasilinear theory.)

We now strengthen this heuristic argument by explicitly deriving $\mathbf{F}_\parallel$ from the dynamical equations of linear tearing modes, for which the reader is referred to Coppi, Greene, and Johnson.[13] We envision the fluctuations to be due to many tearing modes, each resonant on some magnetic surface, and ignore the nonlinear interaction between them. The mean fields now represent equilibrium fields, and the fluctuating fields are the linearized perturbations. For a particular mode with the dependence $\exp[qt+i\mathbf{k}\cdot\mathbf{r}]$, $\mathbf{k}=k\hat{\mathbf{z}}+m\hat{\boldsymbol{\theta}}/r$, the dominant contribution to the average quantity $\mathbf{F}$ comes from the resonance layer $\mathbf{k}\cdot\mathbf{B}_0=0$ where the radial gradients are large. Outside the layer the mode behaves ideally, with zero contribution to $\mathbf{F}$. From the considerations leading to theorem II, it is seen that to leading order $\mathbf{F}\cdot\mathbf{B}_0=-\nabla\cdot\langle(\mathbf{A}_1\cdot\mathbf{B}_0)\mathbf{v}_1\rangle$, which can be shown[13] to equal

$$\mathbf{F}\cdot\mathbf{B}_0=\nabla\cdot\left[\frac{iB_0^2}{\hat{\mathbf{r}}\times\mathbf{k}\cdot\mathbf{B}_0}\langle(\hat{\mathbf{r}}\cdot\mathbf{B}_1)(\hat{\mathbf{r}}\cdot\mathbf{v}_1)\rangle\hat{\mathbf{r}}\right]. \tag{8}$$

To leading order, $\hat{\mathbf{r}}\cdot\mathbf{B}_1$ is constant throughout the layer,[13] and we may rewrite Eq. (8) as $\mathbf{F}\cdot\mathbf{B}_0=-\nabla\cdot[\eta r^{-2}B_{\theta 0}^2\mu' Q\langle\Psi_0\Xi\rangle\hat{\mathbf{r}}]$, where $\mu=rB_{z0}/B_{\theta 0}$, primes denote $r$ derivatives, and the notation is the same as in Ref. 13, where $Q$, $\Psi_0$, and $\Xi$ are the scaled growth rate, the radial components of $\mathbf{B}_1$, and the displacement vector, respectively. The average is now interpreted as an integral over the dimensionless inner-layer variable. Since $\Psi_0=\text{const}$, only the *even* part of $\Xi$ contributes. A solution for the even part of $\Xi$ follows closely the derivation of the odd part in Ref. 13. We finally obtain

$$\mathbf{F}\cdot\mathbf{B}_0=\nabla\cdot[\alpha\eta B_0^2\lambda'\hat{\mathbf{r}}], \tag{9}$$

where $\alpha=Q^{-1/2}\Psi_0^2 f(\mu')$, and $f(\mu')$ is some positive function. $\alpha$ has the dimensions of length squared, and depends on the global equilibrium profile through the mode amplitude $\Psi_0$, and $Q$, both of which depend[13] on $\Delta'$. It is easy to see that $\alpha$ is proportional to $W^4/L^2$, where $W$ is the island width[14] and $L$ is the resistive layer width. To get $\alpha$ of order 1 the mode must certainly be in the nonlinear regime. Nevertheless, our linear analysis yields the correct dependence on $\lambda'$, predicted more generally by the heuristic argument. In a model calculation, such as the one presented later in the paper, $\alpha$ is taken to be $\alpha_0 a^2$, where $\alpha_0$ is a positive constant.

The tearing-mode model may also be used to calculate $\mathbf{F}_\perp$. The calculation, however, is much more tedious and seems unprofitable for linear theory. It is more rewarding to derive an exact expression for $\mathbf{F}$ for nonlinear interchange modes using the similarity method developed in Ref. 9, which we extend and correct as follows. We use local coordinates $(x,y,\theta)$ where $x=(r-r_0)\mu'/r_0$, $y=(z-\mu\theta)/r_0$. Fluctuations vary rapidly in $x$ and $y$ but slowly in $\theta$; that is, $\nabla_\perp \sim O(\delta^{-1})$ and $\nabla_\parallel \sim O(1)$, where $\delta \sim \eta^{1/2}$. The primary correction to Ref. 9 stems from the relation $p_1+\mathbf{B}_0\cdot\mathbf{B}_1=0$ which follows from $O(1)$ terms in the momentum equation $\rho d\mathbf{v}/dt=-\nabla(p+\frac{1}{2}B^2)+\mathbf{B}\cdot\nabla\mathbf{B}$. We thus represent $\mathbf{B}_1=\nabla A_2\times\mathbf{n}-p_1\mathbf{n}$, $\mathbf{v}_1=-\nabla\phi_2\times\mathbf{n}+\Lambda_1\mathbf{n}$, where $\mathbf{n}=\mathbf{B}_0/|\mathbf{B}_0|$. (From now on we drop the subscript 0.) Equations for $A_2$ and $\phi_2$ may be obtained, in the manner of Ref. 9, by collecting $O(\delta)$ terms of the induction and the momentum equations. An inconsistency in the equation for $p_1$ of Ref. 9 may now be cured by deriving it from the projection along $\mathbf{B}_0$ of the equation for $\mathbf{B}_1$. We quote here the main results; details will appear elsewhere. The relevant equations are

$$\frac{d\tilde{A}}{d\tau}=-\frac{\partial\tilde{\phi}}{\partial\theta}+\frac{1}{S}\left(\frac{B}{B_\theta}\right)^2\Delta_\perp\tilde{A}, \tag{10}$$

$$\frac{d}{d\tau}\Delta_\perp\tilde{\phi}=-\frac{\partial}{\partial\theta}\Delta_\perp\tilde{A}-\left(\frac{B}{B_\theta}\right)^2\sigma\left[\frac{\partial\tilde{A}}{\partial x}\frac{\partial}{\partial y}\Delta_\perp\tilde{A}-\frac{\partial\tilde{A}}{\partial y}\frac{\partial}{\partial x}\Delta_\perp\tilde{A}\right]+\beta\frac{B_\theta}{B}\frac{\partial\tilde{p}}{\partial y}, \tag{11}$$

$$\frac{d\tilde{p}}{d\tau}+K\left(\frac{B}{B_\theta}\right)\frac{\partial\tilde{\phi}}{\partial y}=\frac{1}{S}\left(\frac{B}{B_\theta}\right)^2\Delta_\perp\tilde{p}+\sigma\left(\frac{B}{B_\theta}\right)^2\left(\frac{\partial\tilde{\Lambda}}{\partial x}\frac{\partial\tilde{A}}{\partial y}-\frac{\partial\tilde{\Lambda}}{\partial y}\frac{\partial\tilde{A}}{\partial x}\right)-\frac{\partial\tilde{\Lambda}}{\partial\theta}-\frac{4}{\beta}\frac{B_\theta}{B}\frac{\partial\tilde{\phi}}{\partial y}, \tag{12}$$

$$\frac{d\tilde{\Lambda}}{d\tau}=-\frac{\partial\tilde{p}}{\partial\theta}+\left(\frac{B}{B_\theta}\right)^2\sigma\left[\frac{\partial\tilde{p}}{\partial x}\frac{\partial\tilde{A}}{\partial y}-\frac{\partial\tilde{p}}{\partial y}\frac{\partial\tilde{A}}{\partial x}\right]+K\left(\frac{B}{B_\theta}\right)^2\frac{\partial\tilde{A}}{\partial y}, \tag{13}$$

where the notation is the same as that of Ref. 9, with the exception that the natural parameter $\beta=2p/B^2$ replaces the parameter $\beta^+$ of Ref. 9 and $\tilde{\Lambda}=\sqrt{\rho}|\mathbf{B}_0|\Lambda_1/p$. For the resistive $g$ mode, further reduction of Eqs. (10) through (13) is given in Ref. 9. These equations may be shown to be invariant under a family of transformations. As shown in Ref. 9, these transformations determine exactly, to the extent of dimensionless constants, local transport coefficients. The same method yields the expression

$$\mathbf{F}_\perp=-\beta_0\eta\frac{rp'^2}{B^2B_\theta^2\mu'^2}\hat{\mathbf{r}}\times\mathbf{B}, \tag{14}$$

where $\beta_0$ is a constant, and should be positive for $p'<0$ in order to conform with theorem I. We note that $\mathbf{F}_\parallel=0$ for these localized modes, and that we have ignored the radial component of $\mathbf{F}$, which can be determined, of course, but does not contribute to Eq. (2).

We now present a model calculation for the relaxed state in an RFP plasma under the influence of a bath of resistive modes, with tearing modes contributing primarily to $\mathbf{F}_\parallel$ and resistive $g$ modes to $\mathbf{F}_\perp$. The relaxed state is given by $\eta\mathbf{J}-\mathbf{F}=E\hat{\mathbf{z}}$, where the constant $E$ is the applied toroidal voltage per unit length introduced through a fictitious cut in the wall, and $\mathbf{F}$ is given by Eqs. (9) and (14). We thus investigate solutions of the equation

$$\mathbf{J}-\alpha_0\frac{\mathbf{B}}{B^2}\nabla\cdot(a^2B^2\nabla\lambda)+\beta_0\frac{rp'^2}{B^2B_\theta^2\mu'^2}\hat{\mathbf{r}}\times\mathbf{B}=\frac{E}{\eta}\hat{\mathbf{z}}. \tag{15}$$

The pressure is determined for given $\mathbf{B}$ from the equilibrium equation $(p+\frac{1}{2}B^2)'+B_\theta^2/r=0$. We note that the tangential component of $\mathbf{F}$ derived from Eqs. (9) and (14) does not vanish at the wall in contrast with the boundary condition on a perfect conductor. As we have seen, the contribution to $\mathbf{F}$ comes from a thin resistive layer in its vicinity. The layer equations are valid at most a distance of a typical layer width from the wall. We choose not to describe the details of this boundary layer near the wall over which the tangential component of $\mathbf{F}$ should vanish rapidly. The only boundary condition left to impose is $\lambda'=0$ at $r=a$, in conformity with theorem II.

Two observations are in order. First, if $\alpha_0=0$, no solutions with toroidal field reversal are possible. The reason is that the $\theta$ component of Eq. (15) is of the form $B_z'+fB_z=0$, $f\geqslant 0$, which implies that $B_z$ decreases exponentially from the center, but never vanishes. Second, large $\alpha_0$ and $\beta_0$ imply that $\lambda'=O(\alpha_0^{-1})$ and $p'=O(\beta_0^{-1/2})$, and correspond asymptotically to $\mathbf{J}=\lambda_0\mathbf{B}$, with $\lambda_0=\text{const}$. The Woltjer-Taylor state is thus a limiting case of Eq. (15). We note, in reality, that the plasma pressure will be determined by additional loss mechanisms, most importantly by anomalous electron heat conduction and also by radiation, which have to be modeled separately.

The steady-state Eq. (15) was solved numerically, and extensive results will be presented elsewhere. We normalize $B_z$ to the volume-averaged $B_z$ ($\bar{B}_z$), $p$ to $\bar{B}_z^2$, $r$ to the minor radius $a$, and $E$ to $\bar{B}_z\eta/a$. We remark that nonzero values for *both* $\alpha_0$ and $\beta_0$ are in general necessary to obtain profiles in accord with ex-

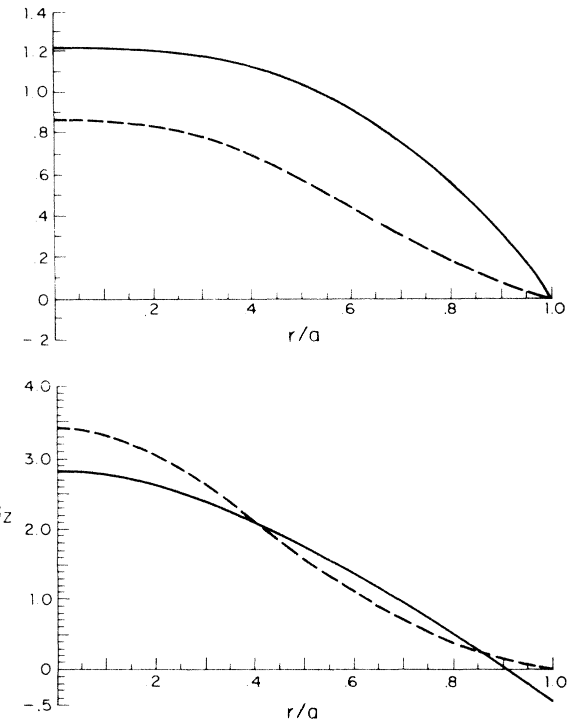


FIG. 1. The pressure and $B_z$ profiles are shown for $\alpha_0$, $\beta_0=0$, 50 (dashed lines) and 20, 50 (solid lines). In both cases $E_0=12$.

perimental observations[2] on ZT-40. While the dominant effect of $\beta_0$ causes $B_z$ to decay exponentially from the center, a finite value of $\alpha_0$ is necessary to explain reversal of $B_z$. This is seen in Fig. 1, where the dashed lines describe the pressure and $B_z$ profiles for $\alpha_0=0$, $\beta_0=50$. The corresponding profiles for $\alpha_0=20$, $\beta_0=50$, represented by solid lines, show reversal of $B_z$ near the edge typical of sustained ZT-40 discharges. In both cases the scaled electric field is $E_0=12$.

We conclude with a summary of the principal results of the paper: (a) the rigorous derivation of theorems I and II, which imply the functional form (1) for $\mathbf{F}_{\parallel}$, (b) dynamical calculations of $\mathbf{F}_{\parallel}$ and $\mathbf{F}_{\perp}$ due to tearing and resistive $g$ modes, respectively, and (c) description of relaxed states in ZT-40 under the influence of **F**. We reiterate that **F** is a dissipative force, has little in common with a conventional "dynamo," but is nonetheless sufficient to explain field reversal in RFP's.

This work was supported by U.S. Department of Energy Grants No. DE-FG02-86ER53222 and No. DE-FG02-86ER53223. A. Bhattacharjee wishes to thank Professor C. K. Chu for his encouragement and support and Y. Baransky for computational aid. Part of the work was done while E. Hameiri was visiting the Eta Beta II experiment in Padua, and he is thankful for the hospitality extended to him.

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