Relativistic Solitary-Wave Solutions of the Beat-Wave Equations

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The relativistic equations governing the nonlinear interaction of two light waves and a Langmuir wave are shown to admit two classes of solitary-wave solutions. *Temporal* solitary waves propagate at speeds greater than the speed of light and carry *no* information. *Spatial* solitary waves propagate at speeds less than the speed of light and *do* carry information. The properties of these waves are discussed and the spatial solitary waves are shown to be well suited to the beat-wave acceleration of particles.

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The nonrelativistic solitary-wave solutions of the three-wave equations[1,2] have been known for some time, and have found applications in many diverse areas of scientific research. An area of current interest is the beat-wave acceleration of particles.[3] One of the problems with the conventional scheme for the beat-wave acceleration of particles is that the width of the Langmuir-wave envelope increases steadily as the wave propagates through the plasma.[4,5] On a time scale of the order of ω_i^{-1}, where ω_i is the ion plasma frequency, ion motion becomes important and destroys the coherence of the Langmuir wave. The resulting turbulent wake drains energy from the incident beams, but cannot be used to accelerate particles. This is a serious waste of incident laser energy. To overcome this difficulty, Mima *et al.*[6] have recently proposed using a solitary-wave structure, in which the Langmuir wave is spatially localized, to accelerate particles.

In order that the particles to be accelerated do not outrun the solitary wave before significant acceleration has been achieved, the speed of the solitary wave must be close to the speed of light. In this sense, the solitary wave under consideration is a highly relativistic space-time modulation of the three-wave envelope functions. The constituent Langmuir wave is weakly relativistic in the sense that the "quiver" velocity associated with the electrostatic field of the wave is much less than the speed of light, so that only the lowest-order corrections to the electron mass need be retained in the equations of motion. This results in a nonlinear reduction in the natural frequency of the Langmuir wave, by an amount which is proportional to the square of the wave amplitude.[7] Neither of these relativistic effects has previously been taken into account for solitary waves.

Originally, Nozaki, Taniuti, and Ohsawa[1,2] derived the nonrelativistic envelope equations in the plasma (or laboratory) frame, and looked for traveling-wave solutions. The solutions they obtained behave nonsingularly as the solitary-wave speed approaches the speed of light. This has led Mima *et al.* to speculate that their solutions are valid for arbitrary solitary-wave speed. One motivation of this Letter is to examine the consistency of this assertion with the special theory of relativity.

The starting point for this investigation is Maxwell's equations, together with the continuity and momentum equations for electrons. If we take advantage of approximations which are valid in the laboratory frame, these are readily combined to give three coupled second-order wave equations. These equations are formulated in terms of the four-potential of each wave, and the linear and nonlinear four-currents, whose Lorentz-transformation properties are well known. It is therefore a simple matter to Lorentz transform the second-order equations to a frame moving with normalized speed β.[8] These are then rewritten approximately as three first-order partial differential equations for the slow temporal and spatial evolution of the wave amplitudes, which are identical in form to the laboratory-frame equations.[5,9] The difference, of course, is that these equations are now formulated in terms of quantities in the moving frame.

These envelope equations admit two classes of solitary-wave solutions. A *temporal* solitary wave is defined to be a wave whose envelope is independent of position in the moving frame. Different points on the wave are related by spacelike intervals and so *no* information can be carried by this type of wave. Conversely, a *spatial* solitary wave is defined to be a wave whose envelope is independent of time in the moving frame. Different points on the wave are related by timelike intervals and so information *can* be carried by this type of wave.

In either case, the governing equations can be written in the canonical form[10]

$$dA_1/d\xi = -is_1A_2A_3, \quad dA_2/d\xi = -is_2A_1A_3^*,$$
$$dA_3/d\xi = -is_3A_1A_2^* + is_3(\delta+\lambda|A_3|^2)A_3. \tag{1}$$

For temporal solitary waves, the dependent variables are the action amplitudes of each wave, the variable ξ represents time, and the s_j's are all equal to unity. For spatial solitary waves, the dependent variables are the action flux amplitude of each wave, the variable ξ represents position, and the s_j's are the signs of the uncoupled group velocities of each wave in the moving frame. Explicit definitions for quantities not defined herein, such as the phase mismatch parameter δ and the nonlinear phase-shift parameter λ, are to be found in Ref. 10 (corresponding definitions for the laboratory-frame equations are to be found in Refs. 5 and 9).

Notice that Eqs. (1) possess the following two invariants:

$$I = s_1|A_1|^2 + s_3|A_3|^2, \quad J = s_2|A_2|^2 - s_3|A_3|^2. \qquad (2)$$

This reflects the fact that the Manley-Rowe relations[11] are satisfied.

Bounded solitary-wave solutions of Eqs. (1) exist, with the property that two of the wave envelopes are spatially localized, providing that any two of s_1, $-s_2$, and $-s_3$ have the same sign. Thus, there are three types of solution to Eqs. (1), classified by which wave is the "pump," i.e., which wave has nonzero amplitude ρ as ξ tends to infinity. For brevity, only those solutions for which wave 1 is the pump wave will be considered explicitly. If we use the invariants (2), the Langmuir-wave amplitude is readily shown to be given by

$$|A_3|^2 = \frac{-2b(1-t^2) + 2[(b^2+4c)(1-t^2)]^{1/2}}{b^2t^2+4c}, \qquad (3)$$

where

$$t = \tanh(a\xi),$$

and

$$a = (4\rho^2 - \delta^2)^{1/2}, \quad b = (4+\delta\lambda)/(4\rho^2-\delta^2),$$

$$c = \lambda^2/[4(4\rho^2-\delta^2)].$$

Once A_3 is known, A_1 and A_2 are found by combining Eqs. (2) and (3). Notice that Eqs. (2) and the "initial" conditions imply that the daughter waves have equal action (flux) densities.

In Fig. 1, the action (flux) amplitude of each wave is plotted as a function of the variable ξ, for different values of δ and λ. The solid line denotes the pump wave, while the broken line denotes the daughter waves. The initial amplitude of the pump wave is equal to -1.0 throughout and ξ is measured in units of the scale length for the case in which δ and λ are both equal to zero. This case is shown in Fig. 1(a). Notice that there is a complete transfer of action between the pump wave and the daughter waves. In general, when the self-nonlinearity of the Langmuir wave is taken into account, the interaction is detuned before a complete transfer of action can take place, as shown in Fig. 1(b). However, if we set the phase-mismatch parameter equal to

$$\delta_{\rm opt} = -\tfrac{1}{2}\lambda\rho^2,$$

the effects of the nonlinear detuning of the interaction can be countered, and a complete transfer of action can take place. This is shown in Fig. 1(c). A similar result was discovered in the conventional beat-wave scheme by Tang, Sprangle, and Sudan.[12] For this value of the phase-mismatch parameter, the speed of the solitary wave is independent of the initial amplitude of the pump wave. However, the scale length of the wave always depends explicitly on the initial amplitude of the pump wave, the phase-mismatch parameter and the speed of the solitary wave.

Having analyzed the solutions of Eqs. (1) in the moving frame, attention is now focused on the form of these solutions in the laboratory frame. The action density n and the action flux density f are the temporal and spatial components, respectively, of the action–flux-density four-vector $(n,f,0,0)$. By use of the transformation laws for four-vectors, it is easily shown

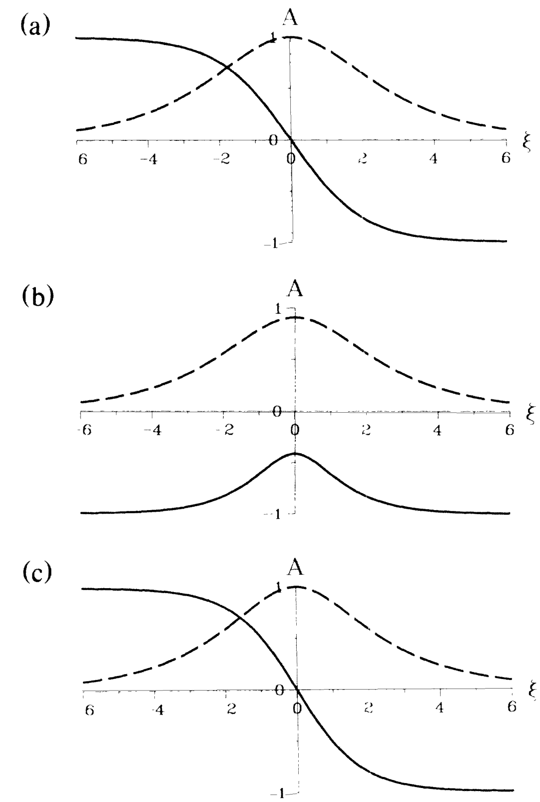

FIG. 1. The action (flux) amplitude of each wave is plotted as a function of position in the moving frame. The solid line denotes the pump wave, while the broken line denotes the daughter waves. The initial amplitude of the pump wave is equal to -1.0 throughout and ξ is measured in units of the scale length for the case in which δ and λ are both equal to zero. (a) $\delta = 0.0$ and $\lambda = 0.0$. (b) $\delta = 0.0$ and $\lambda = 2.0$. (c) $\delta = \delta_{\rm opt} = -1.0$ and $\lambda = 2.0$.

that

$$n_j=\frac{n_j'}{\gamma(1-\beta\beta_j)},\quad f_j=\frac{f_j'}{\gamma(1-\beta\beta_j)},$$

where the β_j's are the group velocities of each wave in the laboratory frame. Henceforth, the convention is adopted that unprimed quantities refer to the laboratory frame, while primed quantities refer to the moving frame. Notice that, as β tends to unity, a finite action (flux) density in the laboratory frame corresponds to an infinite action (flux) density in the moving frame. Rather than transforming the general solution, only the special case in which the phase matching is exact and the self-nonlinearity of the Langmuir wave is neglected will be considered. In this case, the general formula (3) simplifies considerably.

The temporal solutions are

$$n_1(x,t)=\frac{n'(0)}{\gamma(1-\beta\beta_1)}\tanh^2[\mu_t(t-\beta x)],$$
$$n_2(x,t)=\frac{n'(0)}{\gamma(1-\beta\beta_2)}\mathrm{sech}^2[\mu_t(t-\beta x)],\qquad(4)$$
$$n_3(x,t)=\frac{n'(0)}{\gamma(1-\beta\beta_3)}\mathrm{sech}^2[\mu_t(t-\beta x)],$$

where $n'(0)$ is the initial action density of the pump wave in the moving frame. Time is measured in units of c^{-1} and the inverse "period" is given by

$$\mu_t=\left(\frac{\omega_e^2k_3^2\,|v_1/c|^2}{\omega_2\omega_3(1-\beta\beta_2)(1-\beta\beta_3)}\right)^{1/2},$$

where v_1 is the quiver velocity of electrons in the field of the pump wave. It is clear from Eqs. (4) that the temporal solitary waves move with speed $\beta_t=\beta^{-1}$ in the laboratory frame. The tenets of special relativity are not violated by these waves, since they have to be initialized at all points in space (although it is difficult to see how this could be done in practice). Initially, energy is everywhere present in the pump wave and is exchanged only locally with the daughter waves. What *is* remarkable is that when different points on the wave begin to interact, the form of the wave is preserved. Since β is, by construction, less than unity, these temporal solitary waves are constrained to move at speeds greater than the speed of light.

For spatial solitary waves, the solutions are

$$f_1(x,t)=\frac{\beta_1 f'(0)}{\gamma(\beta_1-\beta)}\tanh^2[\mu_s(x-\beta t)],$$
$$f_2(x,t)=\frac{\beta_2 f'(0)}{\gamma(\beta_2-\beta)}\mathrm{sech}^2[\mu_s(x-\beta t)],\qquad(5)$$
$$f_3(x,t)=\frac{\beta_3 f'(0)}{\gamma(\beta_3-\beta)}\mathrm{sech}^2[\mu_s(x-\beta t)],$$

where $f'(0)$ is the initial action flux density of the pump wave in the moving frame and the inverse scale length is given by

$$\mu_s=\left(\frac{\omega_e^2k_3^2\,|v_1/c|^2}{\omega_2\omega_3|\beta_2-\beta|\,|\beta_3-\beta|}\right)^{1/2}.\qquad(6)$$

It is clear from Eqs. (5) that the spatial solitary waves move with speed $\beta_s=\beta$ in the laboratory frame. For the same reason that the temporal solitary waves are constrained to move at speeds greater than the speed of light, the spatial solitary waves are constrained to move at speeds less than the speed of light. This important effect of special relativity is missing from the previous analyses of this three-wave interaction. Using the laboratory-frame equations (which are identical in form to the moving-frame equations[10]), Nozaki[13] has shown that the spatial solitary waves are unstable when the daughter waves have equal group velocities. As β tends to unity, the group velocities of the three constituent waves become equal in the moving frame, *implying that the spatial solitary waves are unstable in this limit.* Notice that the speed of these solitary waves is independent of the initial amplitude of the pump wave. However, the inverse scale length of these solitary waves depends on both the initial amplitude of the pump wave and the speed of the solitary wave. In particular, as β tends to one of the group velocities of the uncoupled waves, the scale length tends to zero and the envelope approximation, upon which the theory of these waves is based, breaks down. This reflects the fact that the uncoupled group velocities delineate the boundaries between the different types of solutions to Eqs. (1), as discussed in Ref. 10.

A remarkable property of the general solitary waves described above is that if the Lorentz parameter β is allowed to be greater than unity in the spatial solutions, the temporal solutions with a Lorentz parameter of β^{-1} are reproduced (and vice versa). This symmetry does not occur in an analysis based on a Galilean transformation of the laboratory-frame equations.

For parameters which are typical of an actual beat-wave accelerator, the lower-frequency light wave plays the role of the pump wave.[10] With δ set equal to its optimum value, the solitary-wave envelope does not differ significantly from the solitary-wave envelope for the case in which the phase matching is exact, and the self-nonlinearity of the Langmuir wave is neglected. Thus the relevant solutions are given by Eqs. (5) and (6), with the subscripts 1 and 2 interchanged.

As a typical example, consider a plasma with an equilibrium density n_0 which is equal to 10^{17} cm^{-3}. This is irradiated by two neodymium-glass lasers, for which ω_2/ω_e, the ratio of the pump frequency to the plasma frequency is equal to 100. The initial quiver velocity of the pump wave, normalized to the speed of light, is equal to 0.1. A spatial solitary wave exists, with speed $\beta_s=0.99$ and peak electrostatic field

$E_{max} = 140$ MeV cm^{-1}. This is larger than the peak accelerating field of conventional accelerators by a factor of 140! If one takes two exponentiations of the amplitude as a measure of the width of the solitary wave, and remembers that the Langmuir-wave envelope is symmetric about the point of maximum amplitude, then it follows from Eq. (6) and Fig. 1(c) that the width of the solitary wave is approximately $130\omega_e^{-1}$. This is sufficiently short that ion motion appears to be unimportant and sufficiently long that the envelope approximation is valid. A test particle, preinjected into the field of the Langmuir wave at a speed close to the speed of light, outruns the solitary wave in a distance of 22 cm. In so doing, it gains 0.79 GeV of energy. Thus, by use of multiple staging, it is theoretically possible to accelerate particles to teraelectronvolt energies in a distance of the order of 1 km.

In the simplified physics of the three-wave model, the spatial solitary waves described herein appear to be well suited to the beat-wave acceleration of particles. However, there are several mathematical and physical questions to be resolved before the practical significance of this scheme can be established. Two of the more important ones are whether the incident laser pulses can be shaped sufficiently to excite these solitary waves, and whether the solitary waves are sufficiently stable to impart a significant amount of energy to the accelerated particles before their demise. These questions are currently under investigation. It is encouraging to note that related solitary-wave excitations have been observed in the Raman-active medium of gaseous nonionized hydrogen,[14,15] corresponding to the $\lambda = 0$ limit of the theory presented here.

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