Nonperturbative Bistability in Periodic Nonlinear Media

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We present a mechanism for a bistability and multistability induced by a periodic modulation of a nonlinear medium. In contrast to usual ones it is nonperturbative with respect to the nonlinearity. The transmitted energy exhibits plateaus as a function of the incident intensity. The frequency versus intensity propagation diagram is fractal. The possible relevance to noise in the conductivity of systems with polarons and to optical, acoustical, or electromagnetic devices is discussed.

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The transmitted intensity of an incident plane wave on a finite one-dimensional nonlinear medium is a nonlinear function of the incident intensity. Such systems can exhibit bistability. Let us consider a piece of nonlinear material $0 \leq x \leq L$ of length L in a stationary regime. An incident plane wave $R_0 e^{ikx}$ on the left ($x \leq 0$) induces a reflected plane wave on the left $R_1 e^{-ikx}$ and a transmitted plane wave Te^{ikx} on the right ($x \geq L$); R_1 and T depend on the wave vector k and on L. The medium is nonlinear; thus the transmission coefficient as a function of the incident intensity $|R_0|^2$ is not a constant. Then, for a given value of the incident intensity, there may be several values of $|R_1|$ and $|T|$; this is called bistability.[1] It is well known that bistability may induce hysteresis in the behavior of the transmitted intensity as the incident intensity is varied.

In the present Letter we describe a new mechanism leading to a bistable behavior. A crucial element in the production of this new mechanism is the presence of a spatial periodic modulation of the medium in addition to the nonlinearity. The effect of a periodic modulation has already been studied from the bistability point of view by Winful, Marburger, and Garmire[2]: There, by a perturbative treatment, enhancement of the bistability is shown to occur in the vicinity of the gaps of the linear theory (for a small periodic modulation and a small nonlinearity). In addition to this type of bistability, we exhibit another type of bistability which cannot be tackled through such a perturbative approach. This new mechanism yields a bistable behavior with a certain number of specific features: (1) The transmitted intensity as a function of the incident intensity exhibits plateaus; (2) the lengths of the plateaus and thus the hysteresis cycles increase with the length of the device; and (3) the passing and nonpassing regions, in a wave number versus transmitted intensity diagram, are fractal.

We first consider for simplicity the stationary discrete nonlinear Schrödinger equation:

$$E\Psi_n = [H\Psi]_n = -\Psi_{n+1} - \Psi_{n-1} + \alpha |\Psi_n|^2 \Psi_n, \quad (1)$$

where n is an integer, $\alpha > 0$ when $0 \leq n \leq L$, and $\alpha = 0$ for $n < 0$ and $N > L$; here the periodic modulation of the medium is provided by the lattice discretization. Equation (1) is an approximate equation for the probability amplitude of an electron in a deformable lattice (polaron). Below we study the similar continuous spatially periodic nonlinear equation (4), which is a common stationary wave equation for the electric field in a spatially modulated nonlinear medium.

Let us look for solutions of the transmission problem associated with (1):

$$\Psi_n = R_0 e^{ikn} + R_1 e^{-ikn}, \quad n \leq 0,$$

$$\Psi_n = Te^{ikn}, \quad n \geq L,$$

with $2\cos(k) = -E$. For a given k, we can solve (1) step by step from $n = L$ to $n = 0$ for a given output Te^{ikn} and then find R_0 and R_1. (Note that changing $|T|$ is the same as varying the nonlinearity parameter α.) If R_0 remains of the same order as T, independently of L, we say that the plane wave with wave number k and outgoing energy $|T|^2$ is in a passing regime. If R_0 appears to be a rapidly increasing function of L (in fact, increasing as $\exp 3^L$), we say that this plane wave is nonpassing. We first investigate numerically (see Fig. 1) the regions in the variables k and

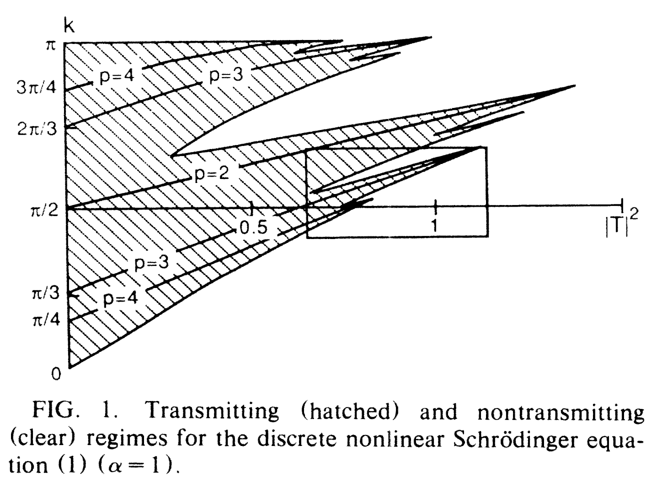

FIG. 1. Transmitting (hatched) and nontransmitting (clear) regimes for the discrete nonlinear Schrödinger equation (1) ($\alpha = 1$).

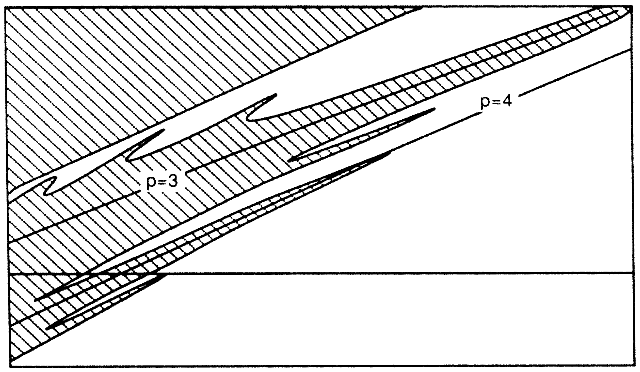

FIG. 2. Enlargement of the region indicated in Fig. 1.

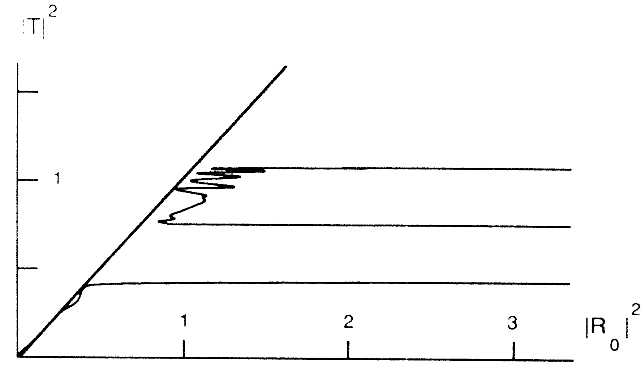

FIG. 3. Transmission diagram of the discrete nonlinear Schrödinger equation.

$|T|^2$ which are passing (hatched) and nonpassing (clear) in the sense described above.

Interesting features undoubtedly appear in this diagram: (1) Several branches (or tongues) appear which are responsible for the bistability and multistability phenomenon as described below; (2) these multiple branches appear only above some threshold in $|T|^2$—we have here a strictly nonperturbative phenomenon (more on this later); and (3) the transmitting region has a fractal shape—we show in Fig. 2 the enlargement of the region indicated in Fig. 1. Actually such a structure appears at any scale.

We turn now to Fig. 3 where, for a given wave number (here $k=2\pi/3$), the transmitted energy $|T|^2$ is plotted versus the incident energy $|R_0|^2$. L has been taken equal to 50. The low-energy part of the curve closely follows the diagonal with some oscillations, corresponding to $|R_0|^2 \approx |T|^2$, that is, to a passing regime. The oscillations are the manifestation of the usual bistability. In this regime, the corresponding point $(k, |T|^2)$ is in a hatched region of Fig. 1. For larger values of $|T|^2$, approximately for $0.45 \leq |T|^2 \leq 0.8$, the plateaus of Fig. 3 correspond to points $(k, |T|^2)$ in a clear region of Fig. 1. The points with larger $|T|^2$ $(0.8 \leq |T|^2 \leq 1.1)$ are again in a hatched region, i.e., in a transmitting regime: The curve of Fig. 3 follows again the diagonal, with stronger oscillations. Finally, we have a second plateau for larger values of $|T|^2$. Thus, for some values of the incident energy R_0 there are several possible values of the transmission, yielding bistability and multistability. Depending on k and on the successive regions met by $(k, |T|^2)$ at fixed k, when $|T|^2$ is increased, this diagram may exhibit various numbers of plateaus, possibly one only. Note that, strictly speaking, and because of the fractal nature of Figs. 1 and 2, a large number $(\approx L)$ of very tiny plateaus exist in the interval of values of $|T|^2$ corresponding to nontransmitting waves. The plateaus seem perfectly flat because $L=50$ is large in view of the divergence rate $(\exp 3^L)$ of R_0; shorter samples give plateaus of shorter lengths. As in other types of bistability there are hysteresis cycles; however, here their lengths depend on the length of the system.

Since Fig. 3 can be deduced from the Fig. 1 through simple considerations, and since Fig. 1 is simpler to explain, we turn now to its theoretical explanation. Equation (1) yields a recursion equation $(\Psi_{n-1}, \Psi_n) = F(\Psi_n, \Psi_{n+1})$, where F is a nonlinear mapping. Thus, for a given wave number k, T being given, the sequence Ψ_n can be computed by application of F repeatedly to the initial condition $(T\exp(ikL), T\exp[ik(L+1)])$. For zero nonlinearity, the solutions of (1) are plane waves, and we explicitly know the trajectories of F, linear in this case. The dynamical system incuded by F is integrable. On the other hand, for finite nonlinearity. F becomes *a priori* a nonintegrable mapping of C^2. In fact, the gauge invariance of (1) allows us to eliminate two variables, corresponding to the arbitrary phase and to the current. We may choose to describe the resulting dynamical system on the plane R^2 in the variables $(|\Psi(n)|^2/|T|^2, |\Psi(n+1)|^2/|T|^2)$: The sequence $(|\Psi(n)|^2/|T|^2, |\Psi(n+1)|^2/|T|^2)$ is the trajectory of some initial condition [namely, (1,1)] by a mapping $(x',y') = f_{k,T}(x,y)$ (which preserves some smooth measure) where $y'=x$, and

$$x' = y + 2(E - \alpha|T|^2 x)(xy - \sin^2 k) + x(E - \alpha|T|^2 x)^2.$$

In general, such a mapping exhibits two kinds of trajectories: quasiperiodic ones and chaotic ones. In our case, no bounded chaotic orbit can be observed, and all the nondiverging trajectories are quasiperiodic.

In Fig. 1, the hatched regions correspond to values of k and T such that the trajectory with initial condition (1,1) is quasiperiodic. In such a situation, the incident intensity R_0 oscillates as L is increased, and remains of the same order as T. On the other hand, in the clear regions, the trajectory diverges like $\exp 3^n$:

(1,1) falls in a chaotic region, its images go to infinity, and R_0, as a function of L, is rapidly increasing.

We are now in a position to understand the important features of Fig. 1. It is known that most of the quasiperiodic orbits of an integrable system (e.g., F for zero nonlinearity) are stable under small perturbations: This explains transmission for the small values of T. On the other hand, in a nonlinear (measure-preserving) dynamical system, possible elliptic periodic points are surrounded by sets of quasiperiodic orbits which form islands around them and that we call here stability basins. We will see that these basins explain the transmitting regime occurring for large values of T in Fig. 1.

At some frequency k, an outgoing wave of intensity T is in a propagating regime if (1,1) is in a stability basin of $f_{k,T}$. For small T, $f_{k,T}$ has an elliptic fixed point near (1,1) (the elliptic fixed point for $T=0$) which has a very large basin so that its basin of stability is likely to contain (1,1): We are in a transmitting regime which can be analyzed in a perturbative framework. For larger values of T, (1,1) may exit this basin but it can enter the stability basin of another periodic point; meanwhile, it may have encountered a chaotic zone. Obviously, this behavior cannot be taken into account by perturbative theories.

So we first describe the couples (k,T) such that (1,1) itself is a periodic point of period p of $f_{k,T}$. *A priori*, an integer p being chosen the equation

$$(1,1)=(f_{k,T})^p(1,1) \tag{2}$$

(two equations for two variables) should have, for dimensional reasons, a discrete set of solutions (k,T). In fact, (1) is invariant under the abscissa reversal, which induces the following symmetry property of $f_{k,T}$:

$$(x',y')=f_{k,T}(x,y) \leftrightarrow (y,x)=f_{k,T}(y',x'). \tag{3}$$

Because of this property, the set of solutions (k,T) of (2) contains a curve. This can be checked easily for the even values $2q$ of the period. Instead of solving (2) with $p=2q$, consider the equation $(1,1)=(f_{k,T})^q(x,x)$, where x may be any real number: This provides two equations for three variables so that the set of appropriate (k,T) is one dimensional. By (3), $(1,1)=(f_{k,T})^q(x,x)$ is equivalent to $(x,x)=(f_{k,T})^q(1,1)$ so that such a (k,T) satisfies $(1,1)=(f_{k,T})^{2q}(1,1)$.

We have computed explicitly these curves for $p=1,2,3$, and 4. For $p=1$, this curve is simply the $T=0$ axis, corresponding to the linear case. For $p=2$ we get the simple curve $|T|^2=-2\cos(k)$; for $|T|^2<\sqrt{2}$, (T,T) is an elliptic two-periodic point, whereas it is hyperbolic for $|T|^2>\sqrt{2}$. Thus this curve (up to $|T|^2=\sqrt{2}$) must lie in the hatched passing region of Fig. 1; furthermore in the neighborhood of this curve the applications $f_{k,T}$ have a two-periodic point near (1,1) and (1,1) is in a stability basin. Correspondingly, we see on Fig. 1 a tongue-shaped transmitting area around the $p=2$ curve, with a width going to zero as $|T|^2$ approaches $\sqrt{2}$ since the size of a stability basin goes to zero as the periodic point changes from elliptic to hyperbolic. In the same way, we have drawn on Fig. 1 the curves for $p=3,4$. This analysis of the smallest periods provides a first good qualitative description of the transmitting area and is enough to emphasize the nonperturbative character of this effect. For any value of p, there exist similar tongues and similar curves starting at $T=0$ and k multiple of π/p. For large p, the corresponding tongues get narrower.

We have described above the apparition of curves of periodic trajectories branching from the curve $T=0$ (corresponding to one-periodic points). In fact, analogous branching curves (coming with their corresponding tongues) appear also starting from any curve corresponding to p-periodic points: This hierarchy is responsible for the fractal character of the frontier between transmitting and insulating regions, as is suggested by Fig. 2.

Note that, as can be understood from the theoretical analysis above, the phenomenon of nonperturbative bistability induced by the lattice does not depend essentially on the type of nonlinearity within large classes of nonlinear interactions and provided strong nonlinearities are allowed. The actual shape of the phase diagram may depend on the specific nonlinearity.

The phenomenon described above for discrete nonlinear wave equations also appears in continuous nonlinear wave equations provided that we add a periodic modulation of the medium. As an example let us consider the nonlinear Schrödinger equation

$$E\Psi=-\Psi''+V\Psi+\alpha|\Psi|^2\Psi, \tag{4}$$

where $V(x)$ is a periodic potential. This is, for instance, the stationary wave equation for the electric field in a nonlinear medium with a periodic modulation of the linear index. The tight-binding approximation of (4) is precisely Eq. (1) and thus the nonperturbative bistability appears also for Eq. (4), in a crossover regime, for periodic potentials yielding a large gap and for small enough sample lengths.

There is an essential difference between the cases $\alpha>0$ and $\alpha<0$: A negative nonlinearity prevents the solutions from diverging as fast as in the discrete case, so that the nonperturbative effect described above can only be a crossover phenomenon. In contrast, a positive nonlinearity allows a strong divergence of the solutions of (4) (in fact, divergence within a finite length). Thus this situation presents a strong analogy with the discrete case, even for small potentials. This

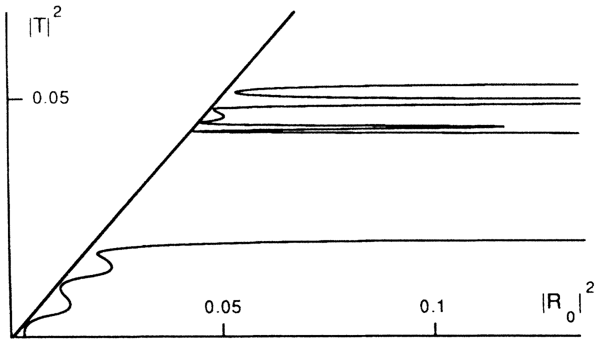

FIG. 4. Transmission diagram of the continuous nonlinear Schrödinger equation with periodic potential.

can be checked numerically: Figure 4 shows a transmission diagram which is analogous to Fig. 3. We have chosen a small periodic potential ($0.2\cos x$) and $E=5$ (i.e., an energy near the middle of the lowest band) in order to obtain two plateaus, separated by a nonpassing regime; for this figure, the sample is 10 periods long. A small change of the sample length slightly modifies the shape of the transmission curve in the passing regime but does not affect the level of the large plateaus. For continuous equations, as in the discrete case, other nonlinear interactions can produce the same type of phenomenon.

Let us turn now to the possible relevance of the nonperturbative bistability induced by a spatial modulation to various domains of physics. Let us first consider the case of semiconductors, in the presence of polarons. The right-hand side of Eq. (1) is a one-dimensional model Hamiltonian for such polarons. On the other hand, for $\alpha=0$, that is, in the linear case corresponding to the treatment of electrons in the tight-binding approximation, the transmission coefficient at the Fermi level is directly related to the electric dc conductance of the system by the Landauer formula[3]: $G=(e^2/h)T/(1-T)$. Assume, that the same relation is valid in the case of polarons. Then Fig. 1 represents regions where the system is insulating or conducting in the diagram variables Fermi level versus nonlinearity. Changing the Fermi level induces large but reproducible fluctuations of the conductance, a phenomenon which has been observed in very narrow quasi one-dimensional structures,[4] but for which other explanations were first proposed. It is tempting to ask whether the effect described in the present paper is responsible for this behavior in some systems and how this theoretical suggestion could be tested experimentally; in particular, we wonder whether one may observe hysteresis in a conductance experiment, as happens in our model. Also note that our effect provides a genuinely nonlinear mechanism for creating a bistable system, that is, a two-level system. Under some noise induced, e.g., by the phonons, it will generate a macroscopic noise; this yields a new mechanism for the generation of noises such as the $1/f$ type.

Let us now come to possible applications of our results to nonlinear optics, acoustics, and electromagnetism. Bistability is clearly an important phenomenon in such fields and is used in many devices. If observable, our new mechanism could prove of interest: By choosing the periodic potential, one could create many different plateaus leading to the coding of potentially as many elementary informations as wished; furthermore, the presence of flat plateaus may be useful for devices such as power-limiting or switching devices. In order to get easily a periodic modulation of the medium it is natural to think of superlattices of semiconductors which could provide appropriate experimental systems to observe and study the nonperturbative bistability. It is also natural to consider modulations of the medium created by a stationary wave (grating), as it is done for the achievement of phase conjugation.

We have not discussed the effect of disorder on the phenomenon described in the present paper; in the case of the usual bistability, we can mention the work of Baylis, Papanicolaou, and White.[5]

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