Accurate Verification of the Conserved-Vector-Current and Standard-Model Predictions

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An approximate analytic calculation of $O(Z\alpha^2)$ corrections to Fermi decays is presented. When the analysis of Koslowksy *et al.* is modified to take into account the new results, it is found that each of the eight accurately studied $\mathcal{F}t$ values differs from the average by $\lesssim 1\sigma$, thus significantly improving the comparison of experiments with conserved-vector-current predictions. The new $\mathcal{F}t$ values are lower than before, which also brings experiments into very good agreement with the three-generation standard model, at the level of its quantum corrections.

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The analysis of the superallowed Fermi transitions has long been one of the cornerstones in the study of the weak interactions. At present it plays a basic role in the verification of the standard model (SM), at the level of its quantum corrections.[1,2] In recent years, experiments in this area have attained great precision ($\simeq 0.1\%$) and, at this level, a sharp discrepancy has been found between the four low-Z and the four high-Z decays.[3,4]

General considerations.—Motivated by the experimental situation and compelling theoretical arguments explained below, we have undertaken the task of reexamining the $O(Z\alpha^2)$ corrections to Fermi decays. These are defined as the residual $O(Z\alpha^2)$ corrections not contained in the product $F(Z,E)(1+\delta_1)$ where $F(Z,E)$ is the Fermi function and δ_1 the $O(\alpha)$ correction.[1] The problem of evaluating the $O(Z\alpha^2)$ as well as $O(Z^2\alpha^3)$ corrections was tackled in the influential papers of Jaus and Rasche[5] (to be called I) and Jaus[6] (to be called II). It was the great merit of these authors to have developed a systematic method to treat these corrections perturbatively, in the framework of the independent-particle model. However, the results obtained in these pioneering papers face, in our opinion, severe difficulties when confronted with theoretical expectations based on general theorems of perturbative quantum field theory. We discuss first salient features of the calculation and results. The crucial diagrams are depicted in Fig. 1. Here q stands for the momentum of the Coulombic photon interacting with the daughter nucleus, while k represents the momentum of the usual four-dimensional photon. There is a second class of diagrams, shown in Fig. 2. The combination of graphs in Fig. 1 is finite in both the ultraviolet and infrared domains. The same is true of the diagrams of Fig. 2 after charge renormalization [which affects Fig. 2(a)] and mass renormalization which must be included in Fig. 2(b). For this reason and the observation that all relevant mass scales are $\ll m_W$, we may treat these $O(Z\alpha^2)$ contributions in the local Fermi theory. According to I, when the daughter nucleus is regarded as a point particle, the graphs in Fig. 1 lead to a large contribution to the decay rate, namely $Z\alpha^2\ln(M/m)$, where M and m are the proton and positron masses, respectively. After the finite nuclear size is taken into account (II), M is essentially replaced by $\Lambda \equiv \sqrt{6}/a$, where a is the rms radius of the charge distribution of the daughter nucleus. As we will see, our own calculations have confirmed the coefficient of the high-frequency logarithm proportional to $\ln M$ or $\ln\Lambda$. Now we pose a theoretical question: Knowing the coefficient of $\ln M$ in the corrections to the total decay rate, can we determine

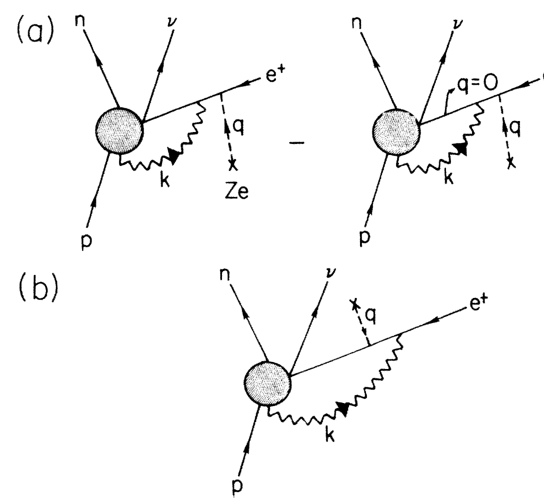

FIG. 1. Leading corrections of $O(Z\alpha^2)$. In (a) one subtracts from the diagram on the left the same diagram with q set equal to zero in the indicated propagator.

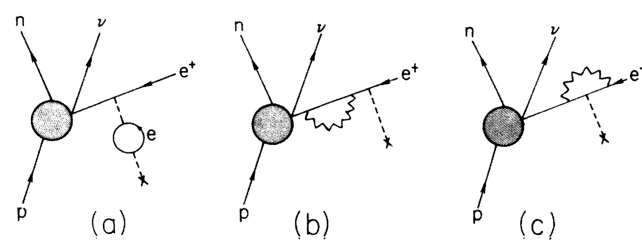

FIG. 2. Nonleading corrections of $O(Z\alpha^2)$

on the basis of general principles the coefficients of the other possible logarithms, $\ln E_m$ and $\ln m$, in the limit $m/E_m \to 0$? (In this paper E and E_m are the relativistic energy of the positron and its maximum value, respectively.) The answer is yes! One simply invokes the theorems on cancellation of mass singularities for total decay rates.[7] If $F(Z,E)$ is expanded in powers of the bare charge e_0, there are no m singularities in the corrections to the total decay rate. Thus, in this case the only m singularities are induced by renormalization. To obtain the coefficient of $\ln m$ we simply expand: $F(Z,E)=1-\pi Z\alpha_0/\beta+\dots$ ($\alpha_0\equiv e_0^2/4\pi$) and insert $\alpha_0=\alpha[1+(2\alpha/3\pi)\ln(1/m)+\dots]$. Clearly, in our case the m singularity is associated with the renormalization of the vacuum-polarization graph of Fig. 2(a). With knowledge of the coefficients of $\ln M$ and $\ln m$, the coefficient of $\ln E_m$ is determined and we reach the conclusion that the corrections to the total decay rate must be of the form

$$\overline{\Delta P}=\bar{P}_0 Z\alpha^2\left[\ln\left(\frac{M}{2E_m}\right)-\frac{2}{3}\ln\left(\frac{2E_m}{m}\right)+c+\dots\right] \quad (1)$$

for a point daughter nucleus and a similar expression with $M\to\Lambda$ when the finite nuclear size is considered. Here the bars over ΔP and P_0 denote integration over the positron spectrum, c is a constant, and the ellipses stand for terms that vanish as $m/E_m\to 0$ as well as very small energy-dependent contributions of $O(E_m/M, E_m/\Lambda)$. Neglecting terms that vanish as $m/E_m\to 0$ corresponds to treatment of the positrons in the extreme relativistic approximation (ERA). Because in all the accurately measured superallowed decays the positron is relativistic throughout most of the spectrum, this is expected to be a good approximation, especially for the high-Z nuclei. The first term in Eq. (1) and part of c arise from Fig. 1, while the second term and the remainder of c derive from Fig. 2. It is very difficult to reconcile the results of (I) and (II) with Eq. (1). The contribution from Fig. 1 was expressed in those papers as $Z\alpha^2[\ln(M/m)+\Delta^0(E_0)]$, where $E_0\equiv E_m/m$. The term $\Delta^0(E_0)$ was evaluated numerically and found to be a positive and *monotonically increasing* function of E_0. Instead, from Eq. (1) one expects that the contributions from Fig. 1 are of the form

$$Z\alpha^2[\ln(M/2E_m)+c'+\dots] = Z\alpha^2[\ln(M/m)-\ln(2E_0)+c'+\dots]$$

and that, therefore, $\Delta^0(E_0)$ should be a *monotonically decreasing* function of E_0. Moreover, unless c' is a large positive constant one expects $\Delta^0(E_0)$ to be negative, especially for the high-Z cases. The determination of c and c' requires a detailed calculation that we now describe.

Detailed calculation.—Our starting point is the consideration of Fig. 1 for a point daughter nucleus. The finite-nuclear-size effect will be discussed later. It will suffice for our purposes to retain only terms of leading order in M because the other contributions involve additional powers of k and q and, as explained in II, are suppressed by small factors of $O(\Lambda/M)$ when the finite nuclear size is considered. After evaluating the two-loop integrals over k and q and performing the appropriate traces we find for the contribution of Fig. 1 at the level of the positron spectrum

$$\frac{\Delta P_1}{P_0}=\frac{2Z\alpha^2 M}{\pi L}\int_0^1\frac{u\,du}{(1-u)}\int_0^1 dv\int_0^1 dz\left[\theta(\hat{A}^2-L^2v^2)\tan^{-1}\left(\frac{Lv}{X}\right)+\theta(L^2v^2-\hat{A}^2)\pi/2+\dots\right], \quad (2)$$

where L is the three-momentum of the positron, $\hat{A}^2\equiv A^2/u(1-u)$, $A^2\equiv\alpha_0^2-L^2(1-uv)^2$, $X\equiv(\hat{A}^2-L^2v^2)^{1/2}$, and, following the notation of I, $\alpha_0\equiv Mu\times v(1-z)+E(1-uv)$. The ellipsis represents subdominant contributions which are not exhibited for brevity. The three-dimensional integrals can be performed analytically in the ERA [followed by the neglect of very small terms of $O(E/M)$]. Including the subdominant contributions we find

$$\Delta P_1=P_0 Z\alpha^2[\ln(M/2E)+5/2-\pi^2/6]. \quad (3)$$

The opposite (nonrelativistic) limit is obtained by our setting $L=0$ in Eq. (2) leading to $\Delta P_1=P_0 Z\alpha^2\times[\ln(M/m)+4-6\ln 2]$. Writing the answer in the form $\Delta P_1/P_0=Z\alpha^2[\ln(M/E)+f(E)]$, we see that $f(E)$ varies from a small positive constant ($\simeq 0.16$) in the extreme relativistic domain to a small negative one ($\simeq -0.16$) in the NR limit. This indicates that $\ln(M/E)$ is by far the dominant contribution. The diagrams from Fig. 2 are much less important numerically. We find in the ERA

$$\Delta P_2=P_0 Z\alpha^2[-\tfrac{2}{3}\ln(2E/m)+\tfrac{1}{6}\pi^2-\tfrac{1}{9}]. \quad (4)$$

Combining (3) and (4) yields

$$\frac{\Delta P}{P_0}=Z\alpha^2\left[\ln\left(\frac{M}{2E}\right)-\frac{2}{3}\ln\left(\frac{2E}{m}\right)+\frac{43}{18}\right]; \quad (5)$$

averaging over the positron spectrum,[8]

$$\left\langle\frac{\Delta P}{P_0}\right\rangle=Z\alpha^2\left[\ln\left(\frac{M}{2E_m}\right)-\frac{2}{3}\ln\left(\frac{2E_m}{m}\right)+\frac{133}{36}\right], \quad (6)$$

or, recalling $E_0\equiv E_m/m$,

$$\left\langle\frac{\Delta P}{P_0}\right\rangle=Z\alpha^2\left[\ln\left(\frac{M}{m}\right)-\frac{5}{3}\ln(2E_0)+\frac{133}{36}\right]. \quad (7)$$

Noting that $\langle \Delta P/P_0\rangle = \overline{\Delta P}/\bar{P}_0$, we see that Eq. (6) is in agreement with the theoretically expected Eq. (1) and furthermore determines c. The constant c' is obtained by the averaging of Eq. (3): $c' = \frac{5}{2} - \pi^2/6 + \frac{47}{60} \simeq 1.64$. Thus $-\ln(2E_0)+c'$ is negative for the eight decays; this leads us to the expectation that the function $\Delta^0(E_0)$ introduced in I should bear the same sign, which confirms the suggestions at the close of our general considerations above. We consider now the effects arising from the finite nuclear size. Neglecting terms of $O(\Lambda/M)$ we find for the fractional correction

$$\Delta = Z\alpha^2\{1-\gamma-\ln(Ma)-4\pi\int_0^\infty \rho(r)r^2\ln(r/a)\,dr\},$$

where $\gamma = 0.5772$ and $\rho(r)$ is the nuclear charge density of the daughter nucleus. We have evaluated Δ for two commonly used distributions: (i)

$$\rho(r) = \text{const}\times(1+\alpha k^2/a^2)e^{-r^2k^2/a^2}$$

with $k^2 = 3(2+5\alpha)/2(2+3\alpha)$ which,[9] with $\alpha = (Z-2)/3$, fits very well several nuclei from ^{4}He to ^{16}O and has been also applied[10] up to $Z=26$; (ii) a uniformly charged sphere of radius R $[a = (\frac{3}{5})^{1/2}R]$. For (i) Δ becomes

$$\Delta = -Z\alpha^2[\ln(M/\Lambda)+\kappa_1(Z)], \quad (8)$$

where

$$\kappa_1(Z) \equiv \tfrac{1}{2}[\gamma+\ln(3/2k^2)+2\alpha/(2+3\alpha)]$$

is a small constant varying from 0.33 for $Z=7$ to 0.36 for $Z=26$. For (ii), Δ is given by an expression analogous to Eq. (8) with $\kappa_1(Z)$ replaced by $\kappa_2(Z) \equiv \gamma - \frac{4}{3} + \frac{1}{2}\ln 10 = 0.40$. The variation in κ is negligible; we will use Eq. (8) with $\kappa_1(Z) = 0.35$ and $a = (\frac{3}{5})^{1/2} r_0 A^{1/3}$ fm (or, equivalently, $M/\Lambda = r_0 A^{1/3}/0.665$); r_0 is given by Wilkinson.[11]

The sum $\langle\delta_2(E)\rangle$ of (7) and (8) is our final expression for the $O(Z\alpha^2)$ corrections[12] and is given numerically in Table I. It ranges from 0.21% for ^{14}O to 0.44% for ^{54}Co while the corresponding correction reported in (II) varies from 0.26% to 0.93%. As explained at the end of our general considerations and in the discussion after Eq. (7) the main difference can be traced to the evaluation of the function $\Delta^0(E_0)$ arising from Fig. 1.[13] A more physical way of characterizing the difference is the following: While papers I and II state that the leading contribution is given by $Z\alpha^2\ln(M/m)$, our analysis indicates that the dominant contribution to the spectrum is roughly $Z\alpha^2\ln(\Lambda/E)$, a significantly different logarithm!

It would be very useful to have an estimate of the corrections of $O(Z^2\alpha^3)$ [defined as the residual contributions not contained in $F(Z,E)(1+\delta_1+\delta_2)$]. The three-loop calculation is by no means trivial. Doing a partial calculation, Jaus managed in II to evaluate the coefficient of the ultraviolet logarithm that arises from the diagrams analogous to Fig. 1 (with one additional Coulombic photon attached to the positron). On the basis of our previous discussion we will identify it as $\ln(\Lambda/E)$. There should exist other logarithms involving m singularities induced by renormalization. Their coefficients can be determined by expansion of $F(Z,E)$ to higher orders in $Z\alpha_0$ and expression of α_0 in terms of α. In an incomplete calculation accompanying mass scales are determined heuristically. A rough estimate that includes the m singularities as well as Jaus's logarithmic term is[14]

$$\delta_3^{\text{HE}} = Z^2\alpha^3[a\ln(\Lambda/E)+bf(E)+dg(E)+h\ln(2E_0)],$$

where

$$a=1.232,\quad b=(4/3\pi)(\tfrac{11}{4}-\gamma-\pi^2/6),\quad d=4/3\pi,\quad f(E)\equiv\ln(2E_m/m)-5/6,$$

$$g(E)\equiv\tfrac{1}{2}[\ln^2(Rm)-\ln^2(2E/m)]+(\tfrac{5}{3})\ln(2RE),\quad R=(\tfrac{5}{3})^{1/2}a,$$

TABLE I. Fractional radiative corrections (in percent) and $\mathcal{F}t$ values.

Decay	$\langle\delta_1^{\text{u}}\rangle$	$\langle\delta_2\rangle$	$\langle\delta_3^{\text{HE}}\rangle$	$\mathcal{F}t$ (s)[a]	$\mathcal{F}t$ (s)[b]
^{14}O	1.29	0.21	0.02	3075.5 ±3.9	3073.4 ±3.9
^{26}Al^m	1.11	0.30	0.04	3072.9 ±3.7	3066.9 ±3.9
^{34}Cl	1.00	0.35	0.06	3076.9 ±4.7	3066.9 ±5.0
^{38}K^m	0.96	0.37	0.07	3076.6 ±4.6	3064.2 ±5.1
^{42}Sc	0.94	0.40	0.08	3089.3 ±7.5	3074.7 ±7.9
^{46}V	0.90	0.41	0.10	3088.6 ±4.3	3071.3 ±5.2
^{50}Mn	0.87	0.43	0.11	3085.9 ±5.7	3066.2 ±6.5
^{54}Co	0.84	0.44	0.12	3087.5 ±4.4	3065.4 ±5.7
Avg.				3080.1 ±2.4	3068.6 ±1.8
CL				3%	80%

[a]Taken from Ref. 4.

[b]Modified values obtained in this paper on the basis of $\langle\delta_1^{\text{u}}+\delta_2\rangle$; $\langle\delta_3^{\text{HE}}\rangle$ has been included as a theoretical error (see text). The theory predicts $\mathcal{F}t$ = const.

TABLE II. Values of V_{ud} based on various combinations of $\mathcal{F}t$ values obtained in this paper. (a) ^{14}O. (b) Average of ^{14}O and ^{26}Al^{m}. (c) Average of four low-Z decays. (d) Average of 8 decays. The last column tests the three-generation SM at the level of its quantum corrections which are $\sim 4\%$. The theory predicts 1.

	V_{ud}	$V_{ud}^2+V_{us}^2+V_{ub}^2$
(a)	0.9740 ±0.0015	0.9971 ±0.0031
(b)	0.9745 ±0.0013	0.9981 ±0.0027
(c)	0.9748 ±0.0011	0.9986 ±0.0023
(d)	0.9747 ±0.0011	0.9984 ±0.0023

and $h=-0.649$ is a constant adjusted to cancel spurious m singularities induced by the factorization $F(Z,E)[1+\delta_1(E)+\ldots]$. The quantity $\langle\delta_3^{\rm HE}(E)\rangle$ (HE means "heuristic estimate") is given numerically in Table I. It will be used as an estimate of the error made in stopping the calculation at $O(Z\alpha^2)$. Table I lists also $\langle\delta_1^{\rm ou}\rangle$, the "outer part" of the $O(\alpha)$ correction.[2,11,15] The "inner corrections" (which depend on m_Z but not E_m) are sizable in the SM[1,2] so that δ_1 is considerably larger than $\delta_1^{\rm ou}$.

Test of conservation of vector currents.—The $\mathcal{F}t$ values from Ref. 4 and the ones obtained by application of the new $O(Z\alpha^2)$ corrections are listed in the last two columns of Table I. We have increased the errors in the new $\mathcal{F}t$ values by combining $\langle\delta_3^{\rm HE}\rangle$ in quadrature with the errors quoted in Ref. 4.[16] We see that the overall pattern is much better than the previous one. Seven of the $\mathcal{F}t$ values differ from the new weighted average by $<1\sigma$, while ^{14}O differs by 1.2σ. The χ^2/ν for the old and new fits are 2.2 and 0.55, respectively, for $\nu=7$ degrees of freedom, corresponding to confidence levels (CL) of 3% and 80%, respectively. If $\langle\delta_3^{\rm HE}\rangle$ is not included in the error, the CL is slightly reduced to 72%. The two lowest-Z values are now further apart, but their average of 3070.2 ±2.8 s is close to the overall average. The weighted averages of the four low-Z and the four high-Z values are now 3068.4 ±2.2 s and 3069.0 ±3.1 s, respectively. Thus, the sharp discrepancy pointed out in Refs. 3 and 4 has disappeared.

Test of the SM.—Table II lists the values of V_{ud} and $V_{ud}^2+V_{us}^2+V_{ub}^2$ based on various combinations of the new $\mathcal{F}t$ values and $V_{us}=0.220\pm0.002$.[17] The treatment of the inner corrections and the calculation of errors is the same as in Ref. 2.[16] It is apparent from Table II that there is now very good agreement between a large number of experiments and the three-generation SM, at the level of its quantum corrections. If the latter were not included, the unitarity bound would be exceeded by $\simeq 3.7\%$ and the SM would not be tenable! In spite of the small errors, the possibility of a fourth generation with substantial mixing[2] is still open. For example, using the last entry in Table II, one finds $|V_{ub'}|\leq 0.064$ (90% CL). It will be interesting to use these calculations to put sharp constraints on various types of new physics, as competing theories must match the SM at a high level of precision. On the experimental side, it would be important to measure accurately ^{10}C[18] and π β decay to verify the consistent trend portrayed in the last column of Table I.

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[18]The ERA is less reliable for low-Z decays such as C but the $O(Z\alpha^2)$ corrections are less important for such cases. It would be desirable to perform new computer-based calculations that retain the small terms neglected in the ERA as well as those of $O(\Lambda/M)$.