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Complexity and the Relaxation of Hierarchical Structures

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We solve exactly the problem of diffusion in an arbitrary hierarchical space. We prove that for a given "tree silhouette" $0 < s < 1$ the dynamic critical exponent ν ranges from $s/(1-s)$, for either uniformly or randomly multifurcating trees, to s for the most diverse ones, in qualitative agreement with a static measure of the tree's complexity. We conclude that uniform trees are optimal for information diffusion, that in thermally activated processes the temperature dependence of ν varies with the underlying tree structure, and that thin elongated trees are the only ones capable of producing a $1/f$ spectrum.

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Many natural and artificial systems have an exact or approximate hierarchical organization.[1] To model the relaxational dynamics of such systems, several authors have recently studied diffusive processes on hierarchical structures.[2–5] Their analysis, however, only applied to the simple case of uniformly multifurcating trees, whose intrinsic self-similarity allows in particular for the use of renormalization-group techniques.[6] But hierarchical structures need not be self-similar. For example, the ground states of the mean-field spin-glass, which possess an ultrametric topology, are known to carry different weights.[7,8] In a more general context, it is precisely the absence of self-similarity that accounts for the complexity of hierarchical structures such as biological organisms and social organizations.[9]

It is therefore desirable to determine how the relaxational dynamics of a hierarchical system depend on its underlying tree structure. This Letter reports the results of such a study. By exactly solving the equation of diffusion in a generic hierarchical space, we have shown that for a given tree silhouette (to be defined below) relaxation is fastest for either uniformly or randomly multifurcating structures and slowest for very diverse ones, in qualitative agreement with a static measure of the tree's complexity.[9] Corollaries of these results include the identification of infinitely thin elongated trees ("brooms") as the only ones capable of producing a $1/f$ spectrum, and the realization that in thermally activated processes, the underlying tree structure can be revealed by the temperature dependence of the dynamic critical exponent ν.

Consider ultradiffusion in an arbitrary hierarchical space. The dynamical equation is given by

$$\frac{dP_i}{dt} = \sum_{j=1}^{N} \epsilon_{ij} P_j, \tag{1}$$

where $i = 1, \ldots, N$ labels the leaves of some arbitrary tree, P_i is the occupation probability for site i, and the hopping rate ϵ_{ij} is only a function of the nearest common ancestor A of i and j on the tree: $\epsilon_{ij} = \epsilon_{ji} = \epsilon_{A(i,j)}$ for $i \neq j$, while $\epsilon_{ii} = -\sum_{j \neq i} \epsilon_{ij}$, thus conserving probability. By appropriately stretching the tree, we can always assume that $\epsilon_A = e^{-h_A}$, where h_A is the height of the branching point A from the bottom of the tree, as shown in Fig. 1(a). We call such trees "metric trees," to stress that not only their topology but also the heights of their branches matter. There is clearly one ultrametric space and one ultradiffusion problem for every metric tree. Note also that there is no loss of generality in having assumed symmetric transition probabilities, since the weight of any leaf can be effec-

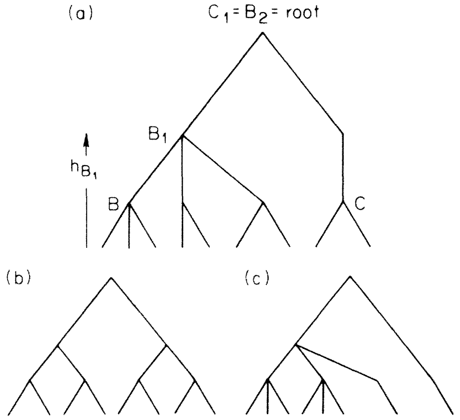

FIG. 1. (a) A generic tree illustrating our notation; the root is the father of B_1 and C and the grandfather of B; B_1 has three sons and seven final descendants. (b) A uniformly bifurcating tree. (c) A very diverse (non-self-similar) tree, with the same silhouette as (b). At each generation the left-half members trifurcate. This tree gives the slowest possible relaxation, as discussed in the text.

tively increased by letting it multifurcate appropriately at low altitude.

We have solved the diffusion equation (1) explicitly, by deriving the complete set of eigenvectors and eigenvalues of the transition matrix ϵ. In order to describe the results succinctly we first introduce some notation and terminology [see also Fig. 1(a)]: For any branch point or tree leaf B, we denote by B_n its unique nth ancestor ($B_0 = B$ by convention, B_1 is the father, B_2 the grandfather, and so on), by N_B the number of final descendants or tree leaves generated by B ($N_B = 1$ if B is itself a leaf), and by S_B the number of immediate offspring or sons of B, when B is a branch point. We also introduce the characteristic function

$$\chi_i(B) = \begin{cases} 1, & \text{if } i \text{ is a descendant of } B \\ 0, & \text{otherwise,} \end{cases}$$

where here $i = 1, \ldots, N$ runs over the leaves. We assume for convenience that branchings may only occur at integral multiples of some minimum adjustable height interval Δh, and will occasionally use the word mth generation for all branches at height $h = m\Delta h$. If $n(h)$ is their total number, we define the silhouette slope

$$s(h) = -\frac{\Delta \ln n(h)}{\Delta h} = \frac{1}{\Delta h}\ln\frac{n(h)}{n(h+\Delta h)},$$

and shall refer to its asymptotic value $s - \lim_{h\to\infty} s(h)$ as the tree's silhouette. Large and small values of s correspond to fat and thin trees, respectively.

Now, for every leaf or branch point B other than the root, there is an eigenvector of ϵ,

$$V_i(B) = N_B^{-1}\chi_i(B) - N_{B_1}^{-1}\chi_i(B_1), \tag{2}$$

which corresponds to the exchange of probability between the descendants of B and those of all his brothers. Its eigenvalue (or inverse characteristic time) depends only on the father B_1:

$$\lambda(B_1) \equiv -\frac{1}{\tau_{B_1}} = -\left[N_{B_1} e^{-h_{B_1}} + \sum_{n=2}^{\text{root}} (N_{B_n} - N_{B_{n-1}}) e^{-h_{B_n}}\right], \tag{3}$$

and is in fact $(S_{B_1} - 1)$-fold degenerate, since the S_{b_1} eigenvectors corresponding to the sons of B_1 obey the linear relation $\sum_{\text{brothers } c} N_c V(c) = 0$.

Finally there is an eigenvector with zero eigenvalue that corresponds to the steady state of equal probability $1/N$ for all sites, and which we denote by $V_i(\text{root})$.

The reader can verify these assertions by working out simple examples; a complete derivation is given in Bachas and Huberman.[10]

For a particle starting out at a leaf l at time zero, the initial condition can be written as

$$P_i(t=0) = \delta_{l,i} = \sum_{n=0}^{\text{root}} V_i(l_n).$$

It then follows that at later times

$$P_i(t) = \sum_{n=1}^{\text{root}} V_i(l_{n-1}) e^{-t/\tau_{l_n}} + \frac{1}{N}. \tag{4}$$

We will be mainly interested in the autocorrelation function, i.e., the probability that the particle returns to its point of departure; by use of Eqs. (2) and (4) it can be written as

$$P_l(t) = \frac{1}{N} + \sum_{n=1}^{\text{root}} \left(\frac{1}{N_{l_{n-1}}} - \frac{1}{N_{l_n}}\right) e^{-t/\tau_{l_n}}. \tag{5}$$

Summing over all initial conditions l, we obtain the average autocorrelation function in the form

$$\bar{P}(t) = \frac{1}{N} + \frac{1}{N} \sum_{\substack{\text{branch points}\\ B}} (S_B - 1) e^{-t/\tau_B}. \tag{6}$$

For finite trees the decay of this function is clearly exponential, and is determined by the smallest nonzero eigenvalue $\lambda(\text{root})$. For infinite trees, however, the asymptotic behavior of $\bar{P}(t)$ is in general modified by the accumulation of eigenvalues near zero. In the remainder of this Letter we shall study the dependence of this asymptotic behavior on the precise structure of the tree.

Consider first a regular uniformly multifurcating tree,[2-5] i.e., one for which every branch at every generation produces b offspring as shown in Fig. 1(b); its silhouette is thus $s = (1/\Delta h)\ln b$. Since $h_{B_n} = h_B + n\Delta h$ and $N_B = b^{h_b/\Delta h}$, we easily deduce from Eqs.

(6) and (3) that

$$\bar{P}(t)=\sum_{m=0}^{\infty}(b-1)b^{-m}\exp\left[-t(be^{-\Delta h})^{m}\left(\frac{e^{\Delta h}-1}{e^{\Delta h}-b}\right)\right]\sim t^{-\nu_{\text{uniform}}},$$

where the dynamic critical exponent

$$\nu_{\text{uniform}}=\ln b/(\Delta h-\ln b)=s/(1-s)\tag{7}$$

depends only on the silhouette which we have assumed lies in the range $0<s<1$. For $s>1$ the eigenvalues of the transition matrix diverge, and relaxation is unstable. Notice, however, that the values $s=0$ and 1 are allowed as asymptotic limits reached from above and below at large h, and leading to logarithmic and stretched exponential decay, respectively.

From Eq. (7) we conclude that among uniform trees the fatter ones relax faster. We must therefore fix the tree's silhouette, in order to study the effect of its structure on dynamics. The following result then shows that stable relaxation is fastest in uniform trees, which are therefore optimal for information diffusion.

Theorem 1.—The dynamic critical exponent of any tree with silhouette $0<s<1$ obeys $\nu\leqslant s/(1-s)$ provided there exists some $w<e$ so that no branch point B has descendants growing faster that w^{h_B-h}.[11]

Sketch of proof.—Using the fact that the average of exponentials is larger than the exponential of the average and that

$$\frac{1}{N}\sum_{B:h_B=h}(s_B-1)=n(h-\Delta h)-n(h)$$
$$=e^{-sh}(e^{s\Delta h}-1),$$

we obtain

$$\bar{P}(t)\geqslant\sum_{m=1}^{\infty}(e^{s\Delta h}-1)e^{-m\Delta hs}e^{-t/\tau_m},\tag{8}$$

where τ_m^{-1} is the average inverse characteristic time of the mth generation that can be bounded from above as follows:

$$\tau_m^{-1}=\frac{\sum(1/t_B)(s_B-1)}{\sum(s_B-1)}\leqslant\exp[m(s-1)\Delta h]\left(\frac{w^{\Delta h}-1}{(e^{s\Delta h}-1)[1-(w/e)^{\Delta h}]}\right),\tag{9}$$

with the summations running over all branch points of the mth generation.

Using inequality (9) in (8), we can finally bound $\bar{P}(t)$ from below by a function with asymptotic power-law decay with exponent ν_{uniform}, Q.E.D.

We next consider random trees, constructed by allowing the multifurcation number x of every branch at every generation to be an independent random variable with probability distribution $p(x)$. The average silhouette is

$$s=(1/\Delta h)\ln\langle x\rangle,$$

where $\langle x\rangle=\sum_{x=1}^{\infty}p(x)x$. It will suffice to average Eq. (5) over all trees, as this automatically takes care of averaging over initial conditions. The trick is to note that $\alpha_n\equiv N_{I_n}/\langle x\rangle^n$, and $\Delta\alpha_\kappa\equiv(N_{I_\kappa}-N_{I_{\kappa-1}})/\langle x\rangle^\kappa$ for $\kappa>n$, (a) are independent random variables, and (b) converge[12] as $n\to\infty$ to random variables α and $\Delta\alpha$ with stationary probability measures $P_1(\alpha)$ and $p(1)\delta(\Delta\alpha)+P_2(\Delta\alpha)$, respectively. Thus the summand on the right-hand side of Eq. (5) becomes, at large n,

$$S_n=\langle x\rangle^{-n}\left\{\langle x\rangle\int_{\langle x\rangle^{-n-1}}^{\infty}\frac{d\alpha}{\alpha}P_1(\alpha)\exp\left[-t\alpha\left(\frac{\langle x\rangle}{e^{\Delta h}}\right)^{(n-1)}\right]\prod_{l>n-1}\left\{p(1)+\tilde{P}_2\left(t\left[\frac{\langle x\rangle}{e^{\Delta h}}\right]^{l}\right)\right\}\right.$$
$$\left.-\int_{\langle x\rangle^{-n}}^{\infty}\frac{d\alpha}{\alpha}P_1(\alpha)\exp\left[-t\alpha\left(\frac{\langle x\rangle}{e^{\Delta h}}\right)^{n}\right]\prod_{l>n}\left\{p(1)+\tilde{P}_2\left(t\left[\frac{\langle x\rangle}{e^{\Delta h}}\right]^{l}\right)\right\}\right\},$$

with $\tilde{P}_2$ the Laplace transform of P_2. If the vanishingly small cutoff of the α integration, which comes from demanding $N_{I_n}\geqslant 1$, could be removed, the summand would obey the homogeneity relation

$$S_{n+1}(t)=e^{-s\Delta h}S_n(t\exp[(s-1)\Delta h]),$$

from which we could deduce that the average autocorrelation function has a power-law decay with exponent $\nu_{\text{random}}=s/(1-s)=\nu_{\text{uniform}}$. Using the integral equation that defines P_1, we have in fact shown that the divergence of the α integration is at most logarithmic,[10] which implies at most $\ln t$ modifications to the above power-law decay. This result can be understood by noticing that both uniform and random trees are balanced, self-similar structures, and hence relax at roughly the same rate.

In order to show that asymptotic diversity, or lack of balance, does actually lead to slower relaxation, we next consider a tree for which the left-half members of every generation trifurcate, while each of the right-half members only gives rise to a single son, as shown in Fig. 1(c). A straightforward calculation[10] then gives a power-law decay with critical exponent $\nu = \ln 2/\Delta h = s$. The following theorem shows that, remarkably, this is the slowest allowed relaxation.

Theorem 2.—The dynamic critical exponent is bounded from below by the silhouette ($\nu \geqslant s$).

Proof.—Since $N_B \geqslant 1$, we have $1/\tau_B \geqslant e^{-h_B}$, and hence

$$\bar{P}(t) \leqslant \textstyle\sum_B (s_B - 1)\exp(-te^{h_B}) = \displaystyle\sum_{n=1}^{\infty} [\exp(s\Delta h) - 1]\exp(-n\Delta hs)\exp(-te^{-n\Delta h}) \sim (1/t)^s,$$

Q.E.D.

We summarize our results schematically in Fig. 2. As can be seen, the dynamic critical exponent is maximized (fastest relaxation) by both uniform and random trees, and minimized (slowest relaxation) by very diverse trees. The same qualitative picture actually obtains if instead of $-\nu$ one plots a static measure of the tree's complexity, or lack of self-similarity, defined by counting the number of nonisomorphic pieces at every generation.[9] This measure should be contrasted with the information-theoretic measure (Shannon's entropy), which is defined by the size of the smallest algorithm that describes how to construct an exact replica of a given tree, and is thus maximized for random trees.[13] A couple of other interesting conclusions follow from our results.

(1) In thermally activated processes $\epsilon_{ij} = \exp(-V_{ij}/kT)$,[11] and hence, if we assume that the structure of the tree is not itself a dynamic variable, rescaling the temperature simply amounts to rescaling all heights (uniformly stretching the tree). Thus s is proportional to T, which implies that $\nu = T/(T_c - T)$ for a uniform tree, while $\nu = T(\ln 2/T_c' \ln 3)$ for the unbalanced tree of Fig. 1(c), where the critical temperatures are in both cases the thresholds above which relaxation becomes unstable. We conclude that the temperature dependence of the critical exponent may reveal the structure of the underlying tree.

(2) In view of Theorem 2, the relaxation of a hierarchical system can have a $1/f$ noise spectrum only if the underlying tree is a "broom," i.e., has vanishing silhouette. Of course, any tree would tend to a broom if infinitely stretched (e.g., by taking $T \to 0$). It is, however, also possible to construct brooms with finite hopping rates, by ensuring that the majority of branches are infertile[10]; these can serve as models for the production of $1/f$ noise at finite temperature or in processes that are not thermally activated.

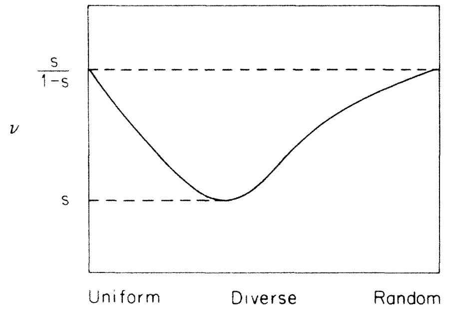

FIG. 2. Schematic plot of the dynamic critical exponent ν vs the Shannon entropy of trees with fixed silhouette s. The broken lines are rigorous upper and lower bounds. A similar plot is obtained if $-\nu$ is replaced by a static measure of complexity that counts nonisomorphic pieces at every level of the tree (Ref. 9).

In summary, we have solved exactly the problem of diffusion in arbitrary hierarchical spaces, and shown that the dynamic critical exponent has the qualitative features required for a measure of complexity. Our study also shows that although asymptotic complexity (or lack of self-similarity) leads to slower relaxation, a $1/f$ spectrum cannot be obtained unless the tree has vanishing silhouette.[14]

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