Three-Dimensional Analysis of Coherent Amplification and Self-Amplified Spontaneous Emission in Free-Electron Lasers

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The growth and saturation of spontaneous emission and coherent radiation in a long undulator are studied by use of the 3D Maxwell-Klimontovich equation. Electron correlation, transverse radiation profiles, spectral features, transverse coherence, and intensity characteristics are discussed. The results, which agree with recent microwave experiments, are applied to proposed schemes for generation of short-wavelength coherent radiation.

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Coherent radiation entering a periodic magnetic structure (an undulator) along with a beam of electrons is amplified by a process which may be called coherent amplification (CA) which is the basis of the free-electron laser (FEL).[1] It is then natural to expect the undulator radiation, the spontaneous emission due to periodic motion of discrete electrons, to be modified by the CA process and, under certain circumstances, to lead to a radiation with different characteristics which may be called self-amplified spontaneous emission (SASE).[2] Qualitative arguments have shown that SASE from long undulators and high-density electron beams could be quite intense, providing a potential source of broadly tunable, coherent radiation at wavelengths below 1000 Å.[3] In this Letter a self-consistent, 3D analysis of SASE is presented.[4] At the same time, the problem of finding in three dimensions an explicit expression for the amplified radiation in terms of the initial amplitude is solved.

It is convenient to choose z, the distance from the undulator entrance, as the independent variable. The transverse coordinates are given by a two-dimensional vector $\mathbf{x}$. The dynamical variables describing the electron motion are the phase θ and the relative energy deviation η. Here θ is roughly the electron coordinates with respect to beam center in units of $\lambda_1/2\pi$, where λ_1 is the radiation wavelength. These variables satisfy the well-known pendulum equations.[5]

To properly analyze SASE, it is important to account for the discreteness of the electrons. This is achieved by utilization of the Klimontovich distribution function $F(\theta,\eta,\mathbf{x};z)$ given by[6]

$$2\pi(n\lambda_1)^{-1}\sum_i \delta(\theta-\theta_i)\delta(\eta-\eta_i)\delta(\mathbf{x}-\mathbf{x}_i).$$

Here, n is the line density of electrons and $\theta_i, \ldots,$ are the instantaneous coordinates of the ith electron. All electrons are assumed to move parallel to the z axis; the effect of electron wiggle is taken into account in the pendulum equations and the generalization to include a beam divergence is discussed later. F is a sum of two parts, $\bar{f}$ and ΔF. Here the background distribution $\bar{f}$ is obtained from F by a two-step averaging process, an ensemble average to remove particle fluctuations and an average over θ to remove the wavelength-scale density modulation arising from the interaction with radiation. The electron beam is assumed to be long, so that end effects can be neglected, and the density is assumed to be uniform in θ. ΔF contains only high-frequency parts responsible for radiation and can be regarded as small compared with $\bar{f}$. Its Fourier transform is

$$h(\nu,\eta,\mathbf{x};z)=(2\pi)^{-1/2}\int d\theta\, e^{i\nu\theta}\Delta F(\theta,\eta,\mathbf{x};z).$$

The radiation field is represented by a complex amplitude $a(\nu,\mathbf{x};z)$, which is the slowly varying part of the full amplitude. ν is the normalized frequency, ω/ω_1, where ω_1 is the resonance frequency given by $2ck_u\gamma_0^2/(1+K^2/2)$. Here γ_0 is the average beam energy in units of mc^2 (m = electron mass, c = velocity of light), $k_u=2\pi/\lambda_u$, λ_u is the undulator period length, and K is the magnetic deflection parameter.[5] The wavelength and wave number corresponding to the frequency ω_1 are denoted by λ_1 and k_1, respectively. The amplitude will have peaks at fundamental frequency ($\nu=1$) and at odd harmonics. Thus it is a good approximation to assume that ν is close to an odd integer, which in the following is taken to be $2l+1$, $l=0,1,\ldots.$ Maxwell's equation, assuming slowly varying amplitude and phase, becomes

$$\left(\frac{\partial}{\partial z}-i\,\Delta\nu\, k_u+\frac{i}{\nu k_1}\frac{\partial^2}{\partial \mathbf{x}^2}\right)a=-\kappa_1\int d\eta\, h. \tag{1}$$

Here $\Delta\nu$ is the frequency shift $\nu-2l-1$, $\kappa_1=eK_l n/4\gamma_0\epsilon_0 ck_1$, ϵ_0 is the vacuum dielectric constant, and $K_l=(-1)^l K[J_l(\xi_l)-J_{l+1}(\xi_l)]$, $\xi_l=(2l+1)K^2/4(1+K^2/2)$, where the J's are Bessel functions.[5]

The continuity equation for the Klimontovich distribution function can be separated into two parts, one describing the high-frequency interaction of h and a, and one describing the slow, nonlinear evolution of $\bar{f}$, as follows:

$$(\partial/\partial z - 2k_u\eta\nu i)h + (e\omega_1/2\gamma_0^2 mc^2)\sum_l K_l a\,\partial\bar{f}/\partial\eta = 0, \tag{2}$$

$$\partial\bar{f}/\partial z + (e/2\gamma_0^2 mcL_b)\sum_l K_l\int d\nu\,\langle a\,\partial h^*/\partial\eta + a^*\partial h/\partial\eta\rangle = 0. \tag{3}$$

In Eq. (3), L_b is the length of the electron beam, the asterisks indicate complex conjugates, and the angular brackets represent the ensemble average. The function $\bar{f}$ varies slowly in z and, for the purpose of Eq. (2), is replaced by the initial distribution $\bar{f}(\eta,\mathbf{x};z=0)=V(\eta)U(\mathbf{x})/\sigma_A$, where $\sigma_A=\int U(\mathbf{x})\,d^2x$ is the cross-sectional area of the electron beam. The functions are normalized by $\int d\eta\; V(\eta)=1$ and $U(0)=1$.

Equations (1) and (2) are linear coupled equations and can be solved by a suitable method. An important parameter characterizing the solution is the dimensionless quantity[2] $\rho=(ne^2K_l^2/32\sigma_A\gamma_0^3k_u^2mc^2\epsilon_0)^{1/3}$, which is typically of order 10^{-3} for the cases considered here. When ρN, where N is the number of the undulator periods, is much smaller than unity, the solution can be expanded in a perturbation series, reproducing the known formula for both undulator radiation and also the small-signal FEL gain. The solution for the more general case is obtained by Van Kampen's methods,[7] which is an eigenfunction expansion applicable to non-Hermitian operators. In the high-gain limit, the resulting expression for the radiation amplitude is

$$a(\nu,\mathbf{x};z)\sim A(\mathbf{x})e^{-2i\mu k_u\rho z}\frac{\int d^2y\,A(\mathbf{y})[a(\nu,\mathbf{y};0)-i\kappa\int d\eta\,h(\nu,\mathbf{y},\eta;0)/T(\mu,\eta)]}{\int d^2y\,A^2(\mathbf{y})[1+U(\mathbf{y})\,dZ(\mu)/d\mu]}, \tag{4}$$

where $\kappa=\kappa_1/2k_u\rho$, $T(\mu,\eta)=\mu+\nu\eta/\rho$, $Z(\mu)=\rho\times\int d\eta\;V'(\eta)/T(\mu,\eta)$, $V'(\eta)=dV(\eta)/d\eta$, and A and μ are, respectively, the eigenfunction and eigenvalue of

$$\left[\mu+\frac{\Delta\nu}{2\rho}-\sigma\frac{\partial^2}{\partial x^2}+U(\mathbf{x})Z(\mu)\right]A(\mathbf{x})=0, \tag{5}$$

where $\sigma=1/2\rho k_1k_u\nu$. Equation (5) is essentially that derived and studied earlier by Moore,[8] except that the effects of momentum spread are included. In one dimension it becomes the dispersion relation studied by several authors.[9] There are, in general, a discrete set of complex eigenvalues, as well as a continuum of real ones. However, the behavior in the high-gain limit is governed by eigenvalue μ with the largest positive imaginary part. In Eq. (4), the term containing $h(\ldots;0)$ describes SASE. The term containing the input amplitude $a(\ldots;0)$ describes CA and represents the solution of the initial-value problem[8] in three dimensions.

To take into account the trajectory excursion due to the electrons' angular spread, it is necessary to consider the properties of electron beam propagation in undulators. Let $\bar{x}(z)$ and $\bar{x}'(z)$ describe the electron trajectory in the absence of FEL interaction with initial conditions $\bar{x}(0)=x$ and $\bar{x}'(0)=x'$. For constant focusing, $\bar{x}(z)=x\cos(k_\beta z)+(x'/k_\beta)\sin(k_\beta z)$ and $\bar{x}'(z)=-xk_\beta\sin(k_\beta z)+x'\cos(k_\beta z)$, where k_β is a constant characterizing the focusing strength known as the betatron wave number. Assume that the phase-space distribution of electrons, $u(x,x')$, is independent of z, i.e., $u(\bar{x}(z),\bar{x}'(z))=u(x,x')$. The eigenvalue equation in this case is similar to Eq. (5) with the last term replaced by

$$-2ik_u\rho\int d\eta\int d^2x'\int_{-\infty}^{0}dz\,e^{\Lambda}\,V'(\eta)\,u(x,x')A(\bar{x}),$$

where

$$\Lambda=-i[2k_u\rho\mu z+2k_u\nu\eta z+(k/2)\int_0^z\bar{x}'(z')^2\,dz']. \tag{6}$$

Although the new eigenvalue equation has not been studied in detail yet, the basic results in this Letter are probably not affected by this generalization.

The transverse behavior in Eq. (4) is completely specified by the mode function $A(\mathbf{x})$. Thus, the radiation in high-gain FEL's is guided, as discussed recently in the literature.[8,10] In addition, it follows from the explicit form of the solution that the radiation is fully coherent transversely. Here, it is assumed that the eigenvalues of Eq. (5) are not degenerate, so that a single mode dominates in the high-gain regime well before saturation. Under certain circumstances, the eigenvalues could become degenerate,[8] and the transverse coherence properties are more complicated. The full transverse coherence is somewhat surprising for SASE and should be compared with the properties of the usual undulator radiation, which is, in general, partially coherent transversely, the degree of coherence being determined by the ratio of radiation to electron-beam phase-space areas.[11] It also follows from the solution that the CA power is maximum when $a(\nu,\mathbf{x};0)\propto A^*(\mathbf{x})$. This means in particular that the curvature of the input phase front is of the same magnitude as but opposite sign to that of the output.

The power is proportional to the ensemble average of $|a|^2$. The interference between the SASE and CA amplitudes clearly vanishes, and the ensemble average of the SASE term can readily be performed on the assumption that electrons are not correlated initially. The intensity growth and the spectral characteristics are mainly determined by the imaginary part μ_I of μ.

Recall that μ is the solution of the eigenvalue equation (5), and is a function of $\Delta\nu$. For the one-dimensional case with zero momentum spread, the maximum value of μ_I is $(\frac{3}{2})^{1/2}$ at $\Delta\nu = 0$. More generally, let the maximum μ_I^m of μ_I occur at $\Delta\nu = \Delta\nu_m$. The growth is then maximum at a frequency given by $\omega_1(2l+1+\Delta\nu_m)$. In general $\Delta\nu_m$ is found to be negative. The behavior of the μ_I about $\Delta\nu_m$ determines the spectral shape. In this way, one obtains the power spectrum,

$$\frac{dP}{d\omega} = e^\tau S(\Delta\omega/\omega_m)\left[g_A\left(\frac{dP}{d\omega}\right)_0 + g_S\frac{\rho E_0}{2\pi}\right], \tag{7}$$

where $\tau = 8\pi\mu_I^m\rho N$, $\Delta\omega = \omega - \omega_m$, $S(x) = \exp(-x^2/2\sigma_\Delta^2)$, and g_A and g_S are quantities of order unity. The first term in Eq. (7) gives the power spectrum for CA, and one finds the growth of the input power spectrum $(dP/d\omega)_0$ to be exponential. The power spectrum for SASE is given by the second term, which exhibits the same exponential growth, with the input replaced by the effective noise power spectrum $\rho E_0/2\pi$, where E_0 is the average beam energy. The function S describes the frequency dependence of the gain for CA, as well as the spectral shape of the SASE radiation. In one dimension, for zero momentum spread, one obtains $g_A = g_S = \frac{1}{9}$ and the bandwidth

$$\sigma_\Delta = (9\rho/2\pi\sqrt{3}N)^{1/2}. \tag{8}$$

For momentum spread much larger than ρ, the eigenvalue μ is real and there is no exponential growth. The total SASE power, obtained by integration over the frequency, is

$$P_{\text{SASE}} = \rho P_{\text{beam}}\, g_S e^\tau / N_{\text{lc}}, \tag{9}$$

where P_{beam} is the kinetic power in the beam (equal to $E_0 I/e$, where I = beam current) and $N_{\text{lc}} = n\lambda_1(2\pi)^{-1/2}/\sigma_\Delta$ is the number of electrons in one coherence length.

From ΔF one obtains information on electron distribution and correlation. For CA the single-particle distribution function develops a coherent modulation. For SASE the modulation occurs in the two-particle correlation function. The correlation, defined as the excess probability of finding two particles compared with the uncorrelated case, is modulated with the periodicity of the radiation wavelength and extends to a distance of one coherence length.

The slow variation of $\bar{f}$ with respect to z is determined by substitution of the solution of the linear equations into Eq. (3)—a procedure known as the quasilinear approximation in plasma physics.[12] From the resulting nonlinear Fokker-Planck equation, one finds that the average value of η must decrease so as to conserve the total energy of the radiation-beam system. In addition, the rms spread σ_η of η is found to increase as $\sigma_\eta^2 \approx \rho^2 g_s e^\tau / N_{\text{lc}}$. Since the growth rate be-

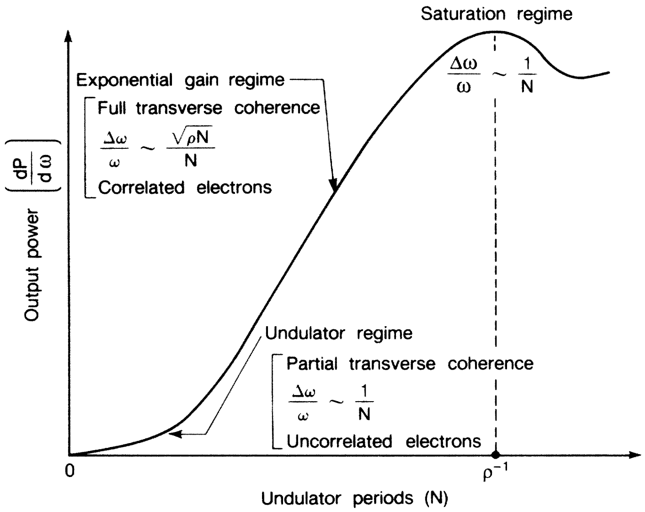

FIG. 1. Schematic representation of SASE characteristics.

comes negligible when $\sigma_\eta >> \rho$, the exponential growth will stop when the factor $g_S e^\tau$ becomes about N_{lc}.[13] The power at saturation becomes, in view of Eq. (9), about ρP_{beam}. For parameters considered here, the saturation occurs at $N \approx 1/\rho$. In view of Eq. (8), the bandwidth at saturation is $\omega/\Delta\omega \sim N$, which is the same as the bandwidth of the spontaneous radiation from an undulator with the same N.

Figure 1 summarizes the characteristics of SASE at different undulator periods N. For $\rho N << 1$, the radiation is an incoherent superposition of radiation from individual electrons, and is referred to as the undulator radiation. It is partially coherent transversely as a result of finite electron-beam emittances. The bandwidth is about $1/N$. For larger N but with $\rho N \lesssim 1$, the FEL interaction causes modulation in the correlation function of electrons, resulting in an enhanced radiation intensity and coherence. Barring certain degenerate situations, the radiation amplitude is dominated by a single mode which is exponentially growing and fully coherent transversely. The relative bandwidth in this exponentially growing regime is smaller than the undulator radiation by a factor $(\rho N)^{1/2}$. Finally, the exponential growth stops when $\rho N \sim 1$ as a result of the increased momentum spread induced by the FEL interaction.

Experimentally, SASE was measured in the microwave region at Lawrence Livermore National Laboratory.[14] For this experiment, the radiation is confined in a waveguide, and therefore 1D theory is appropriate. The data were compared with the prediction given by Eq. (9) in the first paper of Ref. 4. The agreement is encouraging.

A long undulator in a special bypass of an optimized storage ring is a promising SASE source for broadly tunable high-power radiation at short wavelengths.[3] In a recent design of such a system for 400-Å radiation, ρ is about 10^{-3} for a 750-MeV, 200-A electron beam.[15] About 100 MW of transversely coherent power, with a bandwidth of about 10^{-3}, will emerge from an undulator of about 1000 periods.

A more detailed account of this work will be presented elsehwere.

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