Functional Measure for Lattice Gravity

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A procedure is developed for transcription of any measure for the integration over metric fields in the continuum to the Regge-calculus lattice.

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A discrete version of Riemannian geometry and its application to classical Einstein gravity has existed[1] for more than twenty years in the form of Regge calculus. Because of its own intrinsic interest and as a result of its connection with the study of random lattices,[2] this subject has had a revival[3,4] of interest. Such a discrete formulation of gravity theories permits us to consider numerical studies[4] of their quantum counterparts. In addition to a discrete form of the action for such theories we still need a measure for the functional integrals appearing in the Feynman quantization procedure. It is the purpose of this Letter to provide a transcription of a given continuum measure for quantum gravity to the discrete case. The numerical studies referred to previously used *ad hoc* measures. A prescription for the transferring of an integration measure from the continuum to the discrete case will likewise permit a lattice formulation of the Polyakov[5] string theory.

In d space-time dimensions quantum theory is obtained by integrating over the $[d(d+1)/2]$-independent components of the metric tensor $g_{\mu\nu}(x)$. For example the vacuum-to-vacuum amplitude is

$$Z=\int \prod_x \mu(g) \prod_{\mu\leq\nu} dg_{\mu\nu} \exp\{iS[g_{\mu\nu}]\}, \tag{1}$$

where $S[g\mu\nu]$ is an action for gravity and $\mu(g)$ is a continuum measure. Gauge-fixing terms and integrations over ghosts are implied in $\mu(g)$. The reason we have emphasized that what will be presented in this work is a *transcription* of a given continuum measure to the discrete form is that there exist various prescriptions for the continuum measure $\mu(g)$.

(a) DeWitt-Fujikawa[6] measure.—A metric on the deformations of the space-time metric is chosen as

$$\begin{gathered}||\delta g_{\mu\nu}|| = \int d^dx \sqrt{-g}\, G^{\mu\nu\lambda\sigma}\delta g_{\mu\lambda}\delta g_{\nu\sigma}, \\ G^{\mu\nu\lambda\sigma}=[g^{\mu\lambda}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\lambda}-(2/d-C)g^{\mu\nu}g^{\lambda\sigma}], \\ C\neq 0.\end{gathered} \tag{2}$$

Such a metric implies the functional measure

$$\mu(g)=[\det\sqrt{-g}\,G]^{1/2} \approx \prod[-g]^{(d-4)(d+1)/8}. \tag{3}$$

This is also the measure used by Polyakov[5] in his transformation of the bosonic string theory into the conformally invariant quantum Liouville theory.

(b) Konopleva and Popov[7] measure.—This is a scale-invariant measure

$$\mu(g)=[\det G]^{1/2}\approx \prod[-g]^{-(d+1)/2}. \tag{4}$$

(c) Leutwyler[8]-Fradkin and Vilkovisky[9] measure.—This measure follows from a canonical treatment of the gravity problem. It strongly depends on the action to be quantized. For $S[g_{\mu\nu}]=\int d^dx\sqrt{-g}\,[\Lambda-\kappa R]$ the vacuum-to-vacuum amplitude in harmonic coordinates is

$$Z=\int \exp\{iS[g_{\mu\nu}]\}\delta(\partial_\mu(\sqrt{-g}g^{\mu\nu}))\det[\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\delta^\alpha{}_\beta+\cdots]\prod_x\mu(g)\prod_{\mu\leq\nu}dg_{\mu\nu}, \tag{5}$$

with

$$\mu(g)=(g^{00})^{d(d-3)/4}(-g)^{(d^2-5d-8)/8}. \tag{6}$$

An integration over an orbit, under coordinate transformations of a fixed metric $g_{\mu\nu}$, eliminates the coordinate-fixing δ function and the associated Fade'ev-Popov determinant. As on the lattice we do not have to fix further a coordinate system, it is only the $\mu(g)$ of Eq. (6) we wish to transcribe.

In the following part of this work it will prove more convenient to work in the *vielbein* formalism. The *vielbein* $e^\alpha{}_\mu$ is related to the metric tensor $g_{\mu\nu}$ by

$$g_{\mu\nu}(x)=e^\alpha{}_\mu(x)e^\beta{}_\nu(x)\eta_{\alpha\beta}, \tag{7}$$

with $\eta_{\alpha\beta}$ a flat Minkowski metric. In general, lattice calculations are performed in Euclidean space; we shall, however, present our results for the curved space being locally Minkowski. In part this is due to the fact that the transition between a Minkowski and Euclidean formulation of gravity is not as direct as it is for flat metric field theories. The formal transposition of our results to the Euclidean case is straightforward. The relation between the integration over the metric tensor and integrating over the *vielbein* variables is

$$\prod_{\mu\leq\nu} dg_{\mu\nu} = \sqrt{-g}\prod_{\mu,\alpha} de^{\alpha}{}_{\mu}. \tag{8}$$

In Regge calculus[1] curved d-dimensional space-time is approximated by d-dimensional simplices glued together at common $(d-1)$-dimensional subsimplices. Such a configuration is specified by giving all the edge lengths l_{ij} between neighboring vertices i,j. These edge lengths are the dynamical variables of this theory. To obtain the functional measure we found it easiest to work in a *vielbein* formulation. To this end we will develop, within Regge calculus, such formalism.

To each edge (i,j) of a simplex S we assign a flat d-dimensional vector $l^{\alpha}{}_{ij;S}$, $\alpha = 1, 2, \ldots, d$, satisfying

$$l^{\alpha}{}_{ij;S} l^{\beta}{}_{ij;S}\eta_{\alpha\beta} = (l_{ij})^2, \quad \sum_{(ij)\in\Delta} l^{\alpha}{}_{ij} = 0 \tag{9}$$

for all triangles Δ in the simplex S. Within each simplex S we may take the $l^{\alpha}{}_{ij;S}$'s emerging from any vertex as the independent set of *vielbeins* for that simplex. Equation (9) tells us that two vectors $l^{\alpha}{}_{ij;S}$ and $l^{\alpha}{}_{ij;S'}$ associated with the same edge but with different simplices must be related by a Lorentz transformation depending only on the simplices S and S':

$$l^{\alpha}{}_{ij;S'} = [L(S',S)]^{\alpha}{}_{\beta} l_{ij;S}. \tag{10}$$

What is the continuum analog of this relation? The continuum *vielbeins* satisfy

$$D_{\mu} e^{\alpha}{}_{\nu} = -(\omega_{\mu})^{\alpha}{}_{\beta} e^{\beta}{}_{\nu}. \tag{11}$$

$(\omega_{\mu})^{\alpha}{}_{\beta}$ is the spin connection, an infinitesimal Lorentz transformation and D_{μ} is a vector covariant derivative. Translated to the lattice, this covariant derivative is just the difference of the lattice *vielbeins* between two neighboring simplices. The translation of Eq. (11) to the lattice is just Eq. (10).

The transcription of a continuum *vielbein* measure to a lattice is achieved by incorporation of the constraints implied by Eq. (10) and Eq. (11):

$$\prod_{x;\mu,\alpha} de^{\alpha}{}_{\mu}(x) = \left(\prod_{x;\mu,\alpha} de^{\alpha}{}_{\mu}(x)\right)\prod_{x,\mu}\left[\prod_{\alpha,\beta}[(d\omega_{\mu})^{\alpha}{}_{\beta}]\prod_{\nu}\left(\sqrt{-g}\prod_{\gamma}\delta(D_{\mu}e^{\gamma}{}_{\nu} + (\omega_{\mu})^{\gamma}{}_{\delta}e^{\delta}{}_{\nu})\right)\right]. \tag{12}$$

The $\sqrt{-g}$ in the above ensures that the integrations over the spin connections yield a constant independent of the metric.

Noting that Eq. (10) is the lattice version of Eq. (11), the above result may be transcribed to the lattice:

$$\prod_{x;\mu,\alpha} de^{\alpha}{}_{\mu} \to \prod_{S}\prod_{(ij)\in S}\prod_{\alpha} dl^{\alpha}{}_{ij;S}\prod_{\Delta\in S}{}' \delta\left(\sum_{(ij)\in\Delta} l^{\alpha}{}_{ij;S}\right)$$
$$\times \prod_{S,S'} dL(S,S')(l_{SS'}\omega_{SS'})^{(d-1)} \prod_{(i,j)\in S\cap S';\alpha}{}'' \delta(l^{\alpha}{}_{ij;S} - [L(S,S')]^{\alpha}{}_{\beta} l^{\beta}{}_{ij;S}). \tag{13}$$

The single prime indicates that the product is only over $d(d-1)/2$ triangles of the simplex S. The product over simplices S and S' ranges over pairs having a common $(d-1)$-dimensional subsimplex. The $(d-1)$-dimensional hypervolume of this subsimplex is denoted by $\omega_{SS'}$ and the edge dual to it has length $l_{SS'}$. The double prime denotes that this product is over (in view of the first δ function) any $(d-1)$ independent edges (i,j) common to both S and S'. $l_{SS'}\omega_{SS'}$ is one of the lattice analogs[3] of $\sqrt{-g}$. There is one factor $l\omega$ for each edge ij in the final product.

The integration over the Lorentz transformations and over the δ-function constraints may be performed resulting in

$$\prod_{x;\mu,\alpha} de^{\alpha}{}_{\mu} \to \prod_{S}\frac{1}{\mathrm{Vol}(S)}\prod_{(ij)} dl_{ij}\, l_{ij}\prod_{S,S'}(l_{SS'}\omega_{SS'})^{(d-1)}. \tag{14}$$

This is the main result of this work.

We end this Letter with a detailed expression for the measure in two interesting cases. The Leutwyler-Fradkin and Vilkovisky measure in four dimensions is

$$d\mu = \prod_{S}\frac{g^{00}(S)}{[\mathrm{Vol}(S)]^3}\prod_{(ij)} dl_{ij}\, l_{ij}\prod_{SS'}(l_{SS'}\omega_{SS'})^3, \tag{15}$$

where $g^{00}(S)$ is the metric component appropriate to the simplex S; it is expressible in terms of the edge lengths of this simplex. The measure for the Polyakov string theory in two dimensions is

$$d\mu = \prod_{\Delta}\frac{1}{[\mathrm{Area}(\Delta)]^{3/2}}\prod_{(ij)} dl_{ij}(l_{ij})^2\omega_{ij}. \tag{16}$$

In this case ω_{ij} is the length dual to the (ij) edge.

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