Four-Terminal Phase-Coherent Conductance

M. Büttiker
IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10598
(Received 2 June 1986)

A conductance formula for a sample of arbitrary shape with four terminals is derived to describe transport in the limit where carriers can traverse the sample without suffering phase-destroying events. The Onsager-Casimir symmetry relations are deduced. Experiments measure an off-diagonal Onsager coefficient and the magnetoconductance of such a sample is asymmetric even in the presence of an Aharonov-Bohm flux only. Symmetry relations between conductance measurements which exchange the role of current and voltage leads are predicted.

PACS numbers: 72.15.Gd, 73.60.Dt

Experiments on small normal wires and loops[1-3] in the presence of a magnetic field have revealed a magnetoconductance which is asymmetric under magnetic field reversal.[1,2] In these experiments the inelastic or phase-breaking length is large compared to the distance over which the voltage drop is measured. Such an asymmetry is interesting in view of the naive expectation that the measured conductance corresponds to a diagonal Onsager[4,5] coefficient and that therefore the conductance should be symmetric. To be sure, there are a number of effects which can give rise to an asymmetric magnetoresistance,[6] such as a classical Hall contribution to the conductance. However, Büttiker and Imry,[7] analyzing the symmetry of a Landauer[8] conductance formula,[9,10] found that the magnetoresistance can be asymmetric even in the presence of an Aharonov-Bohm flux only. On the other hand, Al'tshuler and Spivak,[11] using a weak-localization approach, maintained that the Onsager relations should hold and that only magnetic impurities could account for the observed asymmetry.

It is the purpose of this Letter to clarify the occurrence of asymmetric magnetoresistances. I propose a Landauer conductance formula which treats the current and voltage terminals in a four-point probe setup explicitly, and on an *equal* footing. Two of the leads carry current to and from the sample and two leads measure the voltage. Previous work on conductance formulas,[8-10,12,13] as far as it was concerned with determining voltage drops over distances short compared to an inelastic length, assumed potentiometers which are "weakly coupled" to the conductor, and matches only currents and not their phase-dependent amplitude at the junction of the conductor and the voltage lead. In contrast, I take into account that the carriers "see" the whole conductor, including the voltage probes, within a phase-breaking length. My result permits the study of symmetries of the conductance under reversal of the role of the current and voltage leads. I show that such a four-terminal conductor obeys the Onsager relations.[4,5] However, experiments measure an off-diagonal Onsager coefficient and the resulting conductance is thus asymmetric under magnetic field reversal. The Onsager relations imply symmetry relations between conductances measured in different lead configurations, known as reciprocity theorems for multipoles.[14] These symmetries are revealed in new experiments[15] specifically undertaken to clarify the asymmetric magnetoresistance.

Consider the conductor shown in Fig. 1. A field dependence is introduced by studying the response of the conductor to an Aharonov-Bohm (AB) flux through the hole.[10,16-18] In a uniform magnetic field,[1-3] there are, in addition to the conductance oscillations with fundamental period[17,18] $\Phi_0 = hc/e$, also aperiodic conductance fluctuations.[1,19] While I focus on the AB oscillations, my conclusions apply equally to the aperiodic conductance fluctuations. The leads in Fig. 1 are connected to reservoirs which are at chemical potentials μ_1, μ_2, μ_3, and μ_4, respectively. The reservoirs serve both as a source and sink of carriers and energy and have the following properties: At zero temperature they feed the leads with carriers up to the energy μ_i. Every carrier coming from the lead and reaching the reservoir is absorbed by the reservoir irrespectively of the phase and energy of the incident carrier. Technically, it is convenient to introduce a piece of perfect wire (unshaded part of the leads in Fig. 1), free of elastic scattering, between the disordered terminals and the reservoirs. First I assume that

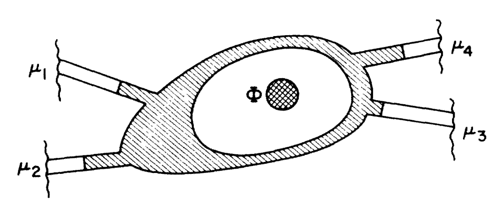

FIG. 1. Disordered normal conductor with four terminals connected via perfect leads (unshaded) to four reservoirs at chemical potentials μ_1, μ_2, μ_3, and μ_4. An Aharonov-Bohm flux Φ is applied through the hole of the sample.

these perfect leads are strictly one-dimensional quantum channels; i.e., there are only two states at the Fermi energy, one with positive velocity (taken to be the direction away from the reservoir) and one with negative velocity. Later I discuss the multichannel case. Scattering in the sample is elastic; inelastic events occur only in the reservoirs. The elastic scattering properties of the sample are described by probabilities $T_{ij}(\Phi)$ for carriers incident in the lead j to be transmitted into the lead i and probabilities $R_{ij}(\Phi)$ for carriers independent in lead i to be reflected into lead i. Current conservation and time-reversal invariance in the presence of a flux imply

$$R_{ii}(\Phi)=R_{ii}(-\Phi), \quad T_{ij}(\Phi)=T_{ji}(-\Phi). \tag{1}$$

Let us determine the currents in the leads. The potentials μ_i are arbitrary within a range at the Fermi energy which is so narrow that the energy dependence of the transmission and reflection probabilities in this range can be neglected. It is convenient to introduce a fifth chemical potential μ_0 which is smaller or equal to the lowest of the four potentials μ_i. Below μ_0 the states with negative and positive velocity are filled and zero net current flows in each of the leads. We need only to consider the energy range $\Delta\mu_i=\mu_i-\mu_0$ above μ_0. The reservoir i injects a current $ev_i(dn_i/dE)\Delta\mu_i$ into the lead i. Here v_i is the velocity at the Fermi energy in lead i, and $dn_i/dE=1/2\pi\hbar v_i$ is the density of states for carriers with negative or with positive velocity at the Fermi energy. Thus the current injected by the reservoir i is $(e/h)\Delta\mu_i$. Consider the current in lead 1. A current $(e/h)(1-R_{11})\Delta\mu_1$ is reflected back to the reservoir 1. Carriers which are injected by the reservoir 2 into lead 2 reduce the current in lead 1 by $-(e/h)T_{12}\Delta\mu_2$ and similarly from the current fed into leads 3 and 4 we obtain in lead 1 a current $-(e/h)(T_{13}\Delta\mu_3+T_{14}\Delta\mu_4)$. Collecting these results and applying similar considerations to determine the currents in other leads yields

$$I_i=\frac{e}{h}\left[(1-R_{ii})\mu_i-\sum_{j\neq i}T_{ij}\mu_j\right]. \tag{2}$$

Note that these currents are independent of the reference potential μ_0 since the coefficients multiplying the potentials add to zero. If we write Eq. (2) in matrix form then both the rows and the columns add to zero.

The first goal is to demonstrate the Onsager relations. Casimir[5] considers a four-pole configuration where a current I_1 is fed into lead 1 and is taken out in lead 3 and a current I_2 is fed into lead 2 and leaves the sample through lead 4. Thus we have to solve Eq. (2) under the condition that $I_1=-I_3$ and $I_2=-I_4$. The result of such a calculation expresses the two currents as a function of differences of voltages, $V_i=\mu_i/e$,

$$I_1=\alpha_{11}(V_1-V_3)-\alpha_{12}(V_2-V_4), \tag{3a}$$

$$I_2=-\alpha_{21}(V_1-V_3)+\alpha_{22}(V_2-V_4). \tag{3b}$$

I find the following expressions for the conductances of Eq. (3):

$$\alpha_{11}=(e^2/h)[(1-R_{11})S-(T_{14}+T_{12})(T_{41}+T_{21})]/S, \tag{4a}$$

$$\alpha_{12}=(e^2/h)(T_{12}T_{34}-T_{14}T_{32})/S, \tag{4b}$$

$$\alpha_{21}=(e^2/h)(T_{21}T_{43}-T_{23}T_{41})/S, \tag{4c}$$

$$\alpha_{22}=(e^2/h)[(1-R_{22})S-(T_{21}+T_{23})(T_{32}+T_{12})]/S, \tag{4d}$$

where

$$S=T_{12}+T_{14}+T_{32}+T_{34}=T_{21}+T_{41}+T_{23}+T_{43}. \tag{5}$$

Taking into account Eq. (1), we see that the diagonal elements are symmetries in the flux $\alpha_{11}(\Phi)=\alpha_{11}(-\Phi)$, $\alpha_{22}(\Phi)=\alpha_{22}(-\Phi)$, and that the off-diagonal elements satisfy $\alpha_{12}(\Phi)=\alpha_{21}(-\Phi)$. Therefore, for the four pole of Fig. 1 the Onsager relations hold. I should emphasize that this is not self-evident: We have a complete *spatial* separation between elastic processes in the conductor and the inelastic processes in the reservoirs. We have introduced irreversibility[16] into the system only by specifying how the reservoirs feed and draw current. Usual derivations of the Onsager relations assume a regression of fluctuations toward a local equilibrium distribution.[4,5]

In a four-terminal experiment only two of the potentials in Fig. 1 are measured! Suppose the current flows from lead 1 to lead 3. The potentials measured are $\mu_2=eV_2$ and $\mu_4=eV_4$ under the condition that the current in leads 2 and 4 is zero. Taking $I_2=0$ in Eq. (3b) yields $V_2-V_4=(\alpha_{22}/\alpha_{21})(V_1-V_3)$ and with use of this in Eq. (3a) the current I_1 can be expressed as a function of V_1-V_3. Thus in this configuration the measured resistance is

$$\mathcal{R}_{13,24}=(V_2-V_4)/I_1=\alpha_{21}/(\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}). \tag{6}$$

Since α_{21} is not symmetric the resistance $\mathcal{R}_{13,24}$ is also not symmetric. This result, however, is completely compatible with the Onsager symmetry relations. The point is that we are measuring an off-diagonal Onsager coefficient and not a diagonal element. It is V_2 and V_4 which determine the voltage drop across the sample and not V_1 and V_3. Now we switch the current and the voltage leads but keep the flux fixed. This means that I_1 in Eq. (3) is zero. This yields a resistance

$$\mathcal{R}_{24,13}=\alpha_{12}/(\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}). \tag{7}$$

The sum of these resistances, $\mathcal{S}_{\alpha}=(\mathcal{R}_{13,24}+\mathcal{R}_{24,13})/2$ is symmetric, as a result of the Onsager relation, $\alpha_{21}(\Phi)=\alpha_{12}(-\Phi)$.

There are two additional possibilities of feeding two currents into the conductor of Fig. 1. Each of these four poles obeys the Onsager relations and the currents and voltages are as in Eq. (3) related by a set of coefficients β and γ. For a given flux we find in general six resistances

$$\mathcal{R}_{mn,kl}=(h/e^2)(T_{km}T_{ln}-T_{kn}T_{lm})/D \tag{8}$$

which differ in magnitude. $D=(h/e^2)^2(\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21})S$ is a subdeterminant of the matrix defined by Eq. (2). All subdeterminants of this matrix are equal and symmetric in the flux. D is independent of the indices $mnkl$. The resistances given by Eq. (2) obey $\mathcal{R}_{mn,kl}=-\mathcal{R}_{mn,lk}=-\mathcal{R}_{nm,kl}$ and the reciprocity relation[14] $\mathcal{R}_{mn,kl}(\Phi)=\mathcal{R}_{kl,mn}(-\Phi)$. The reciprocity relation states that the resistance measured in the presence of a flux Φ is equal to the resistance measured in the presence of a flux $-\Phi$ if the reversal of the flux is accompanied by an exchange in the role of the current and voltage leads. The six resistances which differ in magnitude can be grouped into three pairs, each pair attributed to a four pole. Thus in addition to $\mathcal{S}_{\alpha}$ we also have the combinations $\mathcal{S}_{\beta}=(\mathcal{R}_{14,32}+\mathcal{R}_{32,14})/2$ and $\mathcal{S}_{\gamma}=(\mathcal{R}_{12,43}+\mathcal{R}_{43,12})/2$ which are symmetric in the flux. We can now extend a relation, known in the classical case in the absence of a field,[14] to the present phase-coherent problem in the presence of a flux and show that $\mathcal{S}_{\alpha}+\mathcal{S}_{\beta}+\mathcal{S}_{\gamma}=0$. In the experiments of Ref. 15, which invoke the geometry of Fig. 2, the configuration where current is fed and drawn from the sample at the two leads to the left (or right) of the loop, the resistances are too small to be measured. In our language this implies $\mathcal{S}_{\gamma}\cong 0$ and we have thus the approximate symmetry $|\mathcal{S}_{\alpha}|\cong|\mathcal{S}_{\beta}|$. In this particular geometry the symmetrical part of the measurable resistances are all equal. I emphasize, however, that these symmetrical resistances should not be confused with a diagonal Onsager coefficient. Similarly, the antisymmetric part $\mathcal{A}_{\alpha}=(\mathcal{R}_{12,43}-\mathcal{R}_{43,12})/2$ should not be confused with an off-diagonal Onsager coefficient.

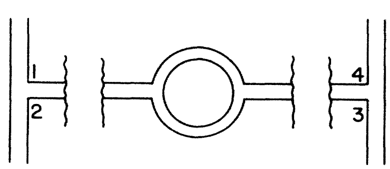

FIG. 2. Loop connected to leads longer than an inelastic scattering length. Phase randomization is introduced by the two reservoirs to the left and right of the loop. The Aharonov-Bohm oscillations of this conductor are symmetric.

The asymmetry discussed here is related to the Hall-type effects discussed elsewhere,[20] but more general since we do not require a particular topology.

Let us generalize these results and assume that the perfect leads have many states at the Fermi energy. In leads of finite cross section we have to consider both the motion of carriers transverse to the lead and the motion along the lead. Motion in the transverse direction is quantized and characterized by a set of discrete energies, E_n. To this energy we have to add the kinetic energy for motion along the direction of the lead, $\hbar^2k^2/2m$, such that, $E_F=\hbar^2k_n^2/2m+E_n$. For each energy E_n which is smaller than E_F we obtain two states at the Fermi energy (quantum channel). Suppose that all the leads are identical and support N quantum channels. The scattering matrix is then a $4N\times 4N$ matrix which can be written in the following form: The probability of a carrier incident in channel n in lead i to be reflected into the same lead into channel m is denoted by $R_{ii,mn}$, and the probability of a carrier incident in lead j in channel n to be transmitted into lead i into channel m is $T_{ij,mn}$. Following Ref. 10 we assume that the reservoir feeds all channels equally up to the chemical potential μ_j. The current injected into each channel is then $(e/h)\Delta\mu_j$ independent of the velocity and the density of states of this channel. The current in lead i due to carriers injected in lead j is $I_{ij}=-(e/h)\sum_{mn}T_{ij,nm}\Delta\mu_j$. Therefore, if we introduce the traces $R_{ii}=\sum_{mn}R_{ii,mn}$, $T_{ij}=\sum_{mn}T_{ij,mn}$, which have the symmetry properties given in Eq. (1), the conductances in the multichannel case are given by Eqs. (4)–(7), except that $1-R_{ii}$ in the single-channel case is replaced by $N-R_{ii}$ in the multichannel case. Thus the symmetry properties of the multichannel case are the same as those discussed above for the single-channel case.

Of particular interest is the implication of these generalized multipole symmetries with regard to Aharonov-Bohm (AB) effect. Because of the AB effect, the resistance, Eq. (6), has a contribution $\Delta\mathcal{R}\cos(2\pi\Phi/\Phi_0-\phi_{\alpha})$, where ϕ_{α} is a sample-specific phase which depends on the particular arrangement of the impurities in the conductor. If we exchange the current and voltage leads, i.e., consider the configuration corresponding to Eq. (7), the resistance, $\mathcal{R}_{24,13}$ must, because of the AB effect, have a contribution, $\Delta\mathcal{R}\cos(2\pi\Phi/\Phi_0+\phi_{\alpha})$, with a phase of opposite sign to that in $\mathcal{R}_{13,24}$. Precisely this phenomenon has now been observed by Benoit *et al.*[15] If we consider the lead configurations corresponding to the measurement of $\mathcal{R}_{14,23}$ and $\mathcal{R}_{23,14}$ they exhibit an AB effect $\Delta\mathcal{R}\cos(2\pi\Phi/\Phi_0\pm\phi_{\beta})$ with a phase ϕ_{β} which is in general different from ϕ_{α}. The AB effect depends only on the flux through the hole of the loop. Thus the experiment confirms the prediction that the resistance can be asymmetric even if only an AB flux is present.

The arbitrary phase ϕ_α or ϕ_β of the Aharonov-Bohm oscillations is a consequence of phase coherence. Suppose the carriers cannot reach the current and voltage leads without suffering an inelastic event. Such a situation can be studied in the model of Fig. 2, where I have introduced reservoirs,[21] with the same properties as discussed above, between the loop and the leads. Each carrier, entering the loop or leaving the loop, has to traverse a reservoir, where the phase and energy are randomized. The total resistance of the conductor is a sum of three terms,[21] corresponding to the resistance of the loop and the resistances of the voltage and current leads. The resistance of the loop is determined by a simple two-terminal resistance formula[22] $\mathcal{R}_0 = (h/e^2)(\mathrm{Tr}\,tt^\dagger)^{-1}$ which is symmetric in the flux, i.e., we obtain a contribution to the total resistance given by[10] $\Delta\mathcal{R}\cos(2\pi\Phi/\Phi_0-\delta)$ with $\delta=0$ or π. The *total* resistance can still be asymmetric, as a result of the classical Hall effect, and the aperiodic fluctuations in the voltage and current leads, but the AB effect will be symmetric in the flux.

The reason that the measured conductance is in general asymmetric is because we measure local potentials and not the potentials which are associated with the reservoirs which serve as the current source and the current sink. This is also the reason for the asymmetry found in Ref. 7. However, the results presented here differ from those of Ref. 7. The difference is due to extra resistances which arise if we couple a voltage lead to a conductor.[21] The asymmetry found in Ref. 7 decreases as the ratio of elastic scattering length to sample length decreases, i.e., as the sample length increases. In contrast, the asymmetry predicted by Eqs. (3)–(7) persists as long as the carriers can traverse the sample coherently. To see this consider the resistance $\mathcal{R}_{13,24}$ for the same topology of Fig. 2. For a long sample we can assume that the probabilities T_{23} and T_{41} which describe transmission through the sample are small compared to the probabilities T_{21} and T_{43} which describe transmission from probe to probe on either side of the sample. In this case the ratio $\mathcal{A}_\alpha/\mathcal{S}_\alpha$ which is a measure of the asymmetry is completely governed by the transmission probabilities T_{21} and T_{43}. The ratio $\mathcal{A}_\alpha/\mathcal{S}_\alpha$ is zero only if T_{21} and T_{43} are symmetric and this is the case only if the probabilities which describe transmission through the sample are strictly zero, i.e., if the length of the conductor exceeds the localization length. I hope that the approach presented here proves useful in the discussion of the size of the conductance fluctuations[1,19] in a four-terminal configuration.

I am indebted to A. D. Benoit, S. Washburn, and R. A. Webb for motivation and discussion of the experimental results.

[1]C. P. Umbach, S. Washburn, R. B. Laibowitz, and R. A. Webb, Phys. Rev. B **30**, 4048 (1984).

[2]R. A. Webb, S. Washburn, C. Umbach, and R. A. Laibowitz, Phys. Rev. Lett. **54**, 2996 (1985); S. Washburn, C. P. Umbach, R. B. Laibowitz, and R. A. Webb, Phys. Rev. **32**, 4789 (1985).

[3]V. Chandrasekhar, M. J. Rooks, S. Wind, and D. E. Prober, Phys. Rev. Lett. **55**, 1610 (1985); S. Datta, M. R. Melloch, S. Bandyopadhyay, R. Noren, M. Vaziri, M. Miller, and R. Reifenberger, Phys. Rev. Lett. **55**, 2344 (1985).

[4]L. Onsager, Phys. Rev. B **38**, 2265 (1931).

[5]H. B. G. Casimir, Rev. Mod. Phys. **17**, 343 (1945).

[6]G. J. C. L. Bruls, J. Bass, A. P. van Gelder, H. van Kempen, and P. Wyder, Phys. Rev. Lett. **46**, 553 (1981).

[7]M. Büttiker and Y. Imry, J. Phys. C **18**, L467 (1985).

[8]R. Landauer, Philos. Mag. **21**, 863 (1970).

[9]M. Ya. Azbel, J. Phys. C **14**, L225 (1981).

[10]M. Büttiker, Y. Imry, R. Landauer, and S. Pinhas, Phys. Rev. B **31**, 6207 (1985).

[11]B. L. Al'tshuler and B. Z. Spivak, Pis'ma Zh. Eksp. Teor. Fiz. **42**, 363 (1985) [JETP Lett. **42**, 447 (1985)].

[12]H.-L. Engquist and P. W. Anderson, Phys. Rev. B **24**, 1151 (1981).

[13]P. W. Anderson, D. J. Thouless, E. Abrahams, and D. S. Fisher, Phys. Rev. B **22**, 3519 (1980); D. C. Langreth and E. Abrahams, Phys. Rev. B **24**, 2978 (1981).

[14]L. J. van der Pauw, Philips Res. Rep. **13**, 1 (1958); H. H. Sample, W. J. Bruno, S. B. Sample, and E. K. Sichel, to be published, treat a magnetic field but assume that the Onsager relations hold locally, i.e., at every point of the sample.

[15]A. D. Benoit, S. Washburn, P. Umbach, R. B. Laibowitz, and R. A. Webb, following Letter [Phys. Rev. Lett. **57**, 1765 (1986)].

[16]B. L. Al'tshuler, A. G. Aronov, and B. Z. Spivak, Pis'ma Zh. Eksp. Teor. Fiz. **33**, 101 (1981) [JETP Lett. **33**, 94 (1981)].

[17]M. Büttiker, Y. Imry, and R. Landauer, Phys. Lett. **96A**, 365 (1983).

[18]Y. Gefen, Y. Imry, and M. Ya. Azbel, Phys. Rev. Lett. **52**, 129 (1984); M. Büttiker, Y. Imry, and M. Ya. Azbel, Phys. Rev. A **30**, 1982 (1984).

[19]A. D. Stone, Phys. Rev. Lett. **54**, 2692 (1985); B. L. Al'tshuler, Pis'ma Zh. Eksp. Teor. Fiz. **41**, 530 (1985) [JETP Lett. **41**, 648 (1985)]; P. A. Lee and A. D. Stone, Phys. Rev. Lett. **55**, 1622 (1985); Y. Imry, Europhys. Lett. **1**, 249 (1986); B. L. Al'tshuler and B. I. Shklovskii, to be published.

[20]M. Büttiker, in *SQUID '85, Superconducting Quantum Interference Devices and Their Applications,* edited by H. D. Hahlbohm and H. Lübbig (Walter de Gruyter, Berlin, New York, 1985), p. 429; O. Entin-Wohlman, C. Hartzstein, and Y. Imry, Phys. Rev. B **34**, 921 (1986); B. L. Al'tshuler and D. E. Khmelnitskii, Pis'ma Zh. Eksp. Teor. Fiz. **42**, 291 (1985) [JETP Lett. **42**, 359 (1985)]; M. Ma and P. A. Lee, to be published, find the same symmetry for the Hall-type effect as Büttiker, in *SQUID '85*, p. 550.

[21]M. Büttiker, Phys. Rev. B **33**, 3020 (1986), and **32**, 1846 (1985).

[22]D. S. Fisher and P. A. Lee, Phys. Rev. B **23**, 6851 (1981).