Renormalization-Group Analysis of Turbulence

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(Received 7 July 1986)

Using renormalization-group methods and the postulated equivalence between the *inertial-range structures* of turbulent flows satisfying initial and boundary conditions and of flows driven by a random force, we evaluate the Kolmogorov constant (1.617) and Batchelor constant (1.161), skewness factor (0.4878), power-law exponent (1.3307) for the decay of homogeneous turbulence, turbulent Prandtl number (0.7179), and von Kármán constant (0.372). This renormalization-group technique has also been used to derive turbulent transport models.

PACS numbers: 47.25.−c

The direct interaction approximation (DIA), due to Kraichnan,[1] was the first field-theoretical approach to the theory of turbulence. Formulated in terms of the Dyson equation, the DIA is characterized as the lowest-order approximation which includes nonlinear corrections to the propagator for the mode $\mathbf{v}(\mathbf{k},\omega)$. It was shown[1] that, in the inertial range, the DIA gives the energy spectrum $E(k)\propto k^{-3/2}$. This result contradicts both experimental data and the Kolmogorov theory of turbulence which gives $E(k)\propto k^{-5/3}$, perhaps with small corrections due to intermittency.

The source of this discrepancy between the DIA and the Kolmogorov theory has long been understood.[2] The DIA does not distinguish between dynamic and kinematic interactions between eddies of widely separated length scales. Small eddies are convected by large eddies in a purely kinematic way which should not lead to energy redistribution between scales. The spurious effect of large-scale convection on small scales has been removed from the DIA by use of a Lagrangean description of the flow. This Lagrangean-history direct interaction approximation[3] (LHDIA) leads to the Kolomogorov 5/3-energy spectrum with the Kolmogorov constant $C_K=1.77$ [see (11) below] which is in reasonable agreement with experiment.[4] However, application of the LHDIA to the problem of turbulent diffusion of a passive scalar does not lead to quantitative agreement with experimental data: The turbulent Prandtl number P_t calculated[4] from the LHDIA is roughly 0.14, much smaller than the experimentally observed $P_t\approx 0.7$–0.9.

In 1977 Forster, Nelson, and Stephen[5] used dynamic renormalization-group (RG) methods, originally developed for the description of the dynamics of critical phenomena,[6] to derive velocity correlations generated by the Navier-Stokes equation with a random-force term. The ideas expressed in Ref. 5 have been used by others in the context of hydrodynamic turbulence.[7–10] The problem is formulated as follows: Consider the d-dimensional space-time Fourier-transformed Navier-Stokes equation for incompressible flow,

$$v_l(\hat{k})=G^0f_l(\hat{k})-\tfrac{1}{2}i\lambda_0G^0P_{lmn}(\mathbf{k})\int v_m(\hat{q})v_n(\hat{k}-\hat{q})\,d^dq/(2\pi)^{d+1}, \tag{1}$$

where the zero-mean Gaussian random force $\mathbf{f}(k,\omega)$ is determined by its correlation function

$$\langle f_i(k,\omega)f_j(k',\omega')\rangle=(2\pi)^{d+1}2D_0k^{-y}P_{ij}(\mathbf{k})\delta(\hat{k}+\hat{k}'). \tag{2}$$

Here

$$G^0=(-i\omega+\nu_0k^2)^{-1},\quad P_{ij}(\mathbf{k})=\delta_{ij}-k_ik_j/k^2,\quad P_{ijk}(\mathbf{k})=k_kP_{ij}(\mathbf{k})+k_jP_{ik}(\mathbf{k}), \tag{3}$$

$\hat{k}=(\mathbf{k},\omega)$, ν_0 is the kinematic viscosity, $\lambda_0=1$, and the constant $y>-2$. The problem (1)–(3) is formulated on the interval $0<k\le\Lambda_0$ and $-\infty<\omega<\infty$, where Λ_0 is a wave number beyond the dissipation wave number at which substantial modal excitations cease. The parameter D_0, which determines the intensity of the random force, is discussed below.

The RG procedure consists of the elimination of modes $\mathbf{v}^>(\hat{k})$ with wave vectors satisfying $\Lambda_0e^{-r}<k<\Lambda$ from the equations of motion for the modes $\mathbf{v}^<(\mathbf{k})$ with wave vectors from the interval $0<k<\Lambda_0e^{-r}$. At this stage, kinematic interactions are excluded by construction and one can expect physically meaningful results in the limit $k\to 0$. Details of this RG procedure are given elsewhere.[5,11]

The RG scale-elimination procedure gives a correction to the bare viscosity ν_0 in terms of an effective viscosity which takes into account the effect of the eliminated modes. The result is

$$\nu_r = \nu_0[1 + A_d\bar{\lambda}_0^2(e^{\epsilon r} - 1)/\epsilon], \quad (4)$$

where $\epsilon = 4 + y - d$, $A_d = \tilde{A}_d S_d/(2\pi)^d$, and

$$\tilde{A}_d = \frac{1}{2}\frac{d^2 - d - \epsilon}{d(d+2)}, \quad S_d = \frac{(2\pi)^{d/2}}{\Gamma(\frac{1}{2}d)}. \quad (5)$$

The dimensionless expansion parameter $\bar{\lambda}_0$ (which is a Reynolds number) is defined as $\bar{\lambda}_0^2 = D_0/\nu_0^3\Lambda_0^\epsilon$. As we shall see below, the choice of $y = d$ recovers the Kolmogorov scaling in the inertial range.

By variation of the cutoff $\Lambda(r) = \Lambda_0 e^{-r}$ we derive differential-recursion relations for $\bar{\lambda}(r) = [D_0/\nu(r)^3 \times \Lambda(r)^3]^{1/2}$ and $\nu(r)$ [11]:

$$\frac{d\nu}{dr} = A_d\nu(r)\bar{\lambda}^2(r), \quad \frac{d\bar{\lambda}^2}{dr} = \bar{\lambda}^2(\epsilon - 3A_d\bar{\lambda}^2). \quad (6)$$

The solutions to (6) are

$$\bar{\lambda}(r) = \bar{\lambda}_0 e^{\epsilon r/2}[1 + 3A_d\bar{\lambda}_0^2(e^{\epsilon r} - 1)/\epsilon]^{-1/2},$$

$$\nu(r) = \nu_0[1 + 3A_d\bar{\lambda}_0^2(e^{\epsilon r} - 1)/\epsilon]^{1/3}.$$

In the limit $r \to \infty$ the coupling parameter $\bar{\lambda}$ (which is an effective Reynolds number) goes to the fixed point

$$\bar{\lambda}_* = (\epsilon/3A_d)^{1/2}$$

and

$$\nu(\Lambda) = (\tfrac{3}{4}A_d D_0)^{1/3}\Lambda^{-\epsilon/3}.$$

Eliminating all modes with $q > k$, we set $\Lambda = k$ and obtain

$$\nu(k) = (\tfrac{3}{4}A_d D_0)^{1/3}k^{-\epsilon/3} \quad (7)$$
$$= 0.4217[2D_0S_d/(2\pi)^d]^{1/3}k^{-4/3}$$

when $y = d = 3$. The coefficient $\tilde{A}_d$ is computed from (5) in the lowest order of ϵ expansion ($\epsilon \to 0$); thus $\tilde{A}_d = 0.2$ in the three-dimensional case $d = 3$.

The energy spectrum can be calculated to lowest order in ϵ from the equation $\mathbf{v}(\hat{k}) = G(\hat{k})\mathbf{f}(\hat{k})$, where the propagator $G(\hat{k})$ is evaluated with the k-dependent viscosity (7). The result is

$$E(k) = 1.186[2D_0S_d/(2\pi)^d]^{2/3}k^{-5/3}. \quad (8)$$

Thus the renormalization-group procedure applied to randomly stirred fluid gives the Kolmogorov spectrum in the case $y = d$.

In order to complete the analysis, it is necessary to relate the parameter D_0 to observables. Consider a fluid described by the Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu_0\nabla^2\mathbf{v}, \quad (9)$$

subject to initial and boundary conditions. We assume that strongly turbulent fluid is characterized in the inertial range of scales by statistically universal scaling laws (Kolmogorov spectrum, etc.) which are independent of initial and boundary conditions. Thus, the system in the universal regime can be described by equations of motion which do not involve any particular initial and boundary conditions; (1) and (2), for example, provided that the random force in (1) and (2) is chosen in such a way that it generates velocity fluctuations which are statistically equivalent to the solutions of (9) subject to initial and boundary conditions. In other words, to describe the fluid in the inertial range we may replace (9) with the corresponding system (1),(2) with a properly chosen force. In this case, it has been shown[12] that if we assume that solutions of Eq. (9) in the inertial range scale as

$$\nu(k) = N\mathscr{E}^{1/3}k^{-4/3} \quad (10)$$

and

$$E(k) = C_K\mathscr{E}^{2/3}k^{-5/3}, \quad (11)$$

then energy balance in analytical turbulence theory requires that the Kolmogorov constant C_K in (11) and the parameter N in (10) be related as

$$N/C_K^2 = 0.1904.$$

Here $\mathscr{E}$ is the rate of energy dissipation in the fluid. Demanding the equivalence of (7) and (8) with (10) and (11) in the inertial range gives

$$2D_0S_d/(2\pi)^d = 1.594\,\mathscr{E}, \quad (12)$$

so that $C_K = 1.617$.

A similar RG procedure[11] applied to the equation of a passive scalar gives the result that the turbulent Prandtl number P_t in the case $y = d = 3$ is

$$P_t^{-1} = \frac{1}{2}\left[-1 + \left(1 + \frac{4(d-1)}{d}\tilde{A}_3^{-1}\right)^{1/2}\right] = 1.3929,$$

so that $P_t = 0.7179$. The Batchelor constant C_{Ba} is defined by the inertial-range scalar fluctuation spectrum. Using energy balance in terms of the k-dependent viscosity at the fixed point, we find[11] $C_{Ba} = C_K P_t$ so that $C_{Ba} = 1.161$. Another calculation[13] of C_{Ba}, based on an RG-modified version of the direct-interaction approximation, gives the same result. The results for the turbulent Prandtl number and the Batchelor constant are in close agreement with experimental data.[14]

The renormalization-group procedure can also be used for deriving averages of different nonlinear operators over the fluctuating velocity field.[11] For example, the skewness factor, which is a dimensionless measure of nonlinear transfer, is defined as

$$\mathscr{S} = -\frac{\langle(\partial v_1/\partial x_1)^3\rangle}{\langle(\partial v_1/\partial x_1)^2\rangle^{3/2}} \equiv \frac{A}{B^{3/2}}, \quad (13)$$

where $\langle \ldots \rangle$ denotes average over the fluctuating velocity field, and

$$A = \langle (\partial v_1/\partial x_1)^3 \rangle = -i \int q_1 p_1 (k-q-p)_1 v_1(\hat{q}) v_1(\hat{p}) v_1(\hat{k}-\hat{q}-\hat{p})\, d^d q\, d^d p/(2\pi)^{2d+2}$$

in the limit $k \to 0$. Decomposing the velocity field into the components $v^<$ and $v^<$ and eliminating small scales using the forced Navier-Stokes equation (1),(2), we find, in the lowest order in the ϵ expansion, that[11]

$$A^< = -i \int q_1 p_1 (k-q-p_1) v_1^<(q) v_1^<(p) v_1^<(\hat{k}-\hat{q}-\hat{p})\, d^d q\, d^d p/(2\pi)^{2d+2}$$
$$= -\tfrac{1}{420}[2D_0 S_d/(2\pi)^d]\,\mathcal{E}/\nu^3\Lambda^2$$

in the limit $k \to 0$ ($r \to \infty$). The same procedure applied to evaluation of B in (13) gives

$$B^< = \frac{1}{20} \frac{2D_0 S_d/(2\pi)^d}{\nu}$$

in the limit $k \to 0$ ($r \to \infty$). Thus

$$\mathcal{S}^<(r) = -\frac{A^<}{(B^<)^{3/2}} = 0.1336 \left\{ \frac{2D_0 S_d/(2\pi)^d}{\nu^3 \Lambda^4} \right\}^{1/2} = 0.4878 \qquad (14)$$

when calculated at the fixed point of the RG calculation. Since $\mathcal{S}^<(r)$ does not depend on r in the limit $r \to \infty$, we assume that (14) holds everywhere in the inertial range, and so $\mathcal{S} = 0.4878$. It should also be noted that the same RG procedure gives the exact result $\mathcal{S} = 0$ in the two-dimensional case $d = 2$.

Another important relation can be derived from the Kolmogorov energy spectrum and formula (7) for the turbulent viscosity. It can be checked readily that the total kinetic energy K in the system is $K = 1.195\mathcal{E}/\nu\Lambda^2$, where Λ is the wave vector corresponding to the integral scale of turbulence. Combining this relation with (7) and (12) we derive a relation between ν, kinetic energy K, and the mean dissipation rate $\mathcal{E}$, namely, $\nu = 0.0837 K^2/\mathcal{E}$.

The RG procedure can be used to evaluate each term of the equations of motion for kinetic energy and dissipation rate. This leads to a so called K - $\mathcal{E}$ model of turbulence. It can be shown[11] that this RG model implies that isotropic turbulence decays as $K \propto (t - t_0)^{-1.3307}$ which is close to the experimental data[14] and recent results of direct numerical simulations.[15] The same model, which does not involve any experimentally adjustable parameters, gives the von Kármán constant[11] $\kappa = 0.372$ for the logarithmic velocity profile.

The good agreement of the RG-predicted constants (C_K, C_{Ba}, P_t, $\mathcal{S}$, κ) with experimental data is to some extent surprising since the RG procedure does not take into account local interactions between eddies of similar size. However, it has been pointed out[9] that the ratio of time constants which correspond to nonlocal and local interactions is $O(\epsilon^{1/2})$. Thus, local interactions are weak if ϵ is assumed small. It remains to be explained why the lowest-order truncation of the RG expansion in powers of $\epsilon = 4$ works so well.

We are grateful to Dr. A. Yakhot and Dr. W. Dannevik for numerous suggestions which influenced the course of this work. We would also like to acknowledge support of this work by the U. S. Air Force Office of Scientific Research under Contract No. F49620-85-C-0026, the U. S. Office of Naval Research under Contracts No. N00014-82-C-0451 and No. N00014-85-K-0201, and the National Science Foundation under Grants No. MSM-8514128 and No. ATM-8414410.

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