Role of Modular Invariance in Evaluation of Gauge and Gravitational Anomalies in the Heterotic-String Theory

Hisao Suzuki[(a)] and Akio Sugamoto
National Laboratory for High Energy Physics (KEK), Tsukuba, Ibaraki 305 Japan
(Received 14 July 1986)

It is known that anomalies can be expressed as a quantity in a short-time limit of the proper time (Schwinger parameter) in point-particle field theories. A natural extension of these expressions for the anomalies is given in the heterotic-string theory. It is shown that the gauge and gravitational anomalies in the heterotic-string theory can be expressed as a surface integral over the Schwinger parameter. The modular invariance of hexagon anomalies allows us to restrict the region of integration to the boundary of the fundamental region. This invariance also shows that the surface integral vanishes, i.e., hexagon gauge and gravitational anomalies are absent in the heterotic-string theory.

PACS numbers: 11.17.+y, 12.10.Gq

Heterotic-string theory[1] is now considered as one of the most promising candidates for the unified theory including gravity. We expect this theory to be finite to all orders of perturbation.[2] It is indeed finite at one-loop order.[3] The mechanism of finiteness in closed-string theories is very different from those in point-particle theories. The loop amplitude of a closed-string theory is derived in the form[4]

$$\int \frac{d^2\tau}{(\mathrm{Im}\tau)^2} F(\tau,\tau), \tag{1}$$

where the domain of integration is the following primitive region

$$P\colon\quad -\tfrac{1}{2} \le \mathrm{Re}\tau \le \tfrac{1}{2},\quad \mathrm{Im}\tau \ge 0. \tag{2}$$

If the amplitude is invariant under the modular transformation

$$\tau \rightarrow (a\tau+b)/c\tau+d), \tag{3}$$

where a,b,c,d are integers satisfying $ad-bc=1$, we are allowed to restrict the region of integration from P to the fundamental region. One of the fundamental regions is given by

$$F\colon\quad \tfrac{1}{2} \le \mathrm{Re}\tau \le \tfrac{1}{2},\quad \mathrm{Im}\tau \ge 0,\quad |\tau| > 1, \tag{4}$$

in which the amplitude is expected to be finite except for the infrared region. The infrared singularity must be more seriously considered than that in point-particle theories, since this singularity is connected with the ultraviolet singularity by modular transformation. The singularity is absent in theories free from tachyonic states. Therefore, type-II superstring theory[5] and heterotic-string theory can be finite. In other fundamental regions, (1) takes a value identical to that given in F because of the modular invariance. The observation of modular invariance requires a certain modification of the field-theoretical treatment based on point-particle theories. So far we have discussed the amplitudes and their finiteness, but it is unclear how to deal with other quantum-theoretical effects such as anomalies in order to be reconciled with the invariance under the modular transformation. Our purpose is to provide one such prescription in the case of anomalies, which can be considered as a natural extension of the treatment in point-particle field theories. A basic observation in our treatment is that the loop amplitude in point-particle theories can be expressed as an integral over the proper time, whereas the anomaly is expressed as a short-time limit of this parameter.[6] What we propose in this paper is to provide an extension of this observation to closed-string theories. Let us consider an anomaly given as a surface integral over the Schwinger parameter,

$$\int_{\partial P} d\tau^* F(\tau,\tau). \tag{5}$$

We will postulate here the modular invariance of expression (5). Then we can reduce the integration region to the boundary of the fundamental region F. Because of the invariance under the modular transformation, (5) takes an identical value for any other fundamental region.

Moreover, the reduction of (5) to an integral over the surface of the fundamental region leads to the following result: The surface integral (5) over the fundamental region vanishes. The line integral consists of three parts: the integrals over lines $\mathrm{Re}\tau = -\frac{1}{2}$, $\mathrm{Re}\tau = \frac{1}{2}$, and $|\tau| = 1$. The first line integral cancels exactly the second one because of the invariance $\tau \rightarrow \tau+1$. Vanishing of the integral over $|\tau|=1$ can be obtained by the modular transformation $\tau \rightarrow -1/\tau$, which maps this line onto itself in the opposite direction. Therefore, if an anomaly is given in the form (5) and turns out to be modular invariant, the anomaly is absent. As we will see later, the gauge and gravitational anomalies in the heterotic string can be expressed in the form (5).

We will consider the parity-violating part of the six-point amplitude in the heterotic string with external gauge and gravitational fields:

$$\int d^{10}p \int \left(\prod_{i=0}^{5} \frac{d^2 z_i}{\pi |z_i|^2} \right) \mathrm{Tr} \left[\Gamma_{11} \prod_{i=0}^{5} F_0 z_i^{\tilde{L}_0^*} V_{R_2}(k_i, 1) \right], \tag{6}$$

where the vertex operator and L_0, $\tilde{L}_0$, F_0 are given in terms of string coordinates

$$X^\mu = \frac{1}{2} x^\mu + \frac{1}{2} p^\mu \frac{i}{2} \ln z + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu z^n, \qquad \tilde{X}^\mu = \frac{1}{2} x^\mu + \frac{1}{2} p^\mu \frac{i}{2} \ln z^* + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu z^{n*}, \tag{7}$$

Ramond's Γ^μ

$$\Gamma^\mu(z) = \gamma^\mu + i\sqrt{2} \gamma_{11} \sum_{n \neq 0} d_n^\mu z^n, \tag{8}$$

and ghosts c_n, e_n, as follows:

$$\begin{aligned} L_0 &= \tfrac{1}{8} p^2 + \sum_{n=1}^{\infty} \alpha_{-n}^\mu \alpha_n^\mu + \sum_{n=1}^{\infty} n d_{-n}^\mu d_n^\mu + \sum_{n=1}^{\infty} n (c_{-n}^* c_n + c_{-n} c_n^*) + \sum_{n=1}^{\infty} n (e_{-n}^* e_n + e_{-n} e_n^*), \\ F_0 &= \frac{-i}{2\sqrt{2}} (\gamma \cdot p) + \gamma_{11} \left[\sum_{n \neq 0}^{\infty} \alpha_{-n}^\mu d_n^\mu - 2 \sum_{n=-\infty}^{\infty} c_{-n}^* e_n + \tfrac{1}{2} \sum_{n=-\infty}^{\infty} n c_{-n} e_n^* \right], \\ \tilde{L}_0 &= \tfrac{1}{8} p^2 + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\mu + \tfrac{1}{2} P^2 + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^I \tilde{\alpha}_n^I - 1 + \sum_{n=1}^{\infty} n (\tilde{c}_{-n}^* \tilde{c}_n + \tilde{c}_{-n} \tilde{c}_n^*), \end{aligned} \tag{9}$$

and

$$\begin{aligned} V_{R_2}(k,z) &= [\eta_\mu^a \Gamma^\mu(z) T^a + \eta_{\mu\nu} \Gamma^\mu(z) \tilde{P}^\nu(z^*)] \exp\{ ik \cdot [X(z) + \tilde{X}(z^*)] \} \\ &= [\eta_\mu^a \Gamma^\mu(z) T^a + \eta_{\mu\nu} \Gamma^\mu(z) \tilde{P}^\nu(z^*)] V_0(k,z). \end{aligned} \tag{10}$$

The Γ_{11} is chosen in order to anticommute with Γ^μ as

$$\Gamma_{11} = \gamma_{11} \exp\left\{ i\pi \left[\sum_{n=1}^{\infty} (d_{-n}^\mu d_n^\mu + e_{-n}^* e_n - e_{-n} e_n^*) + e_0 e_0^* \right] \right\}. \tag{11}$$

The matrix T^a is defined by

$$T = \begin{cases} \tilde{P}^I(z=1) = P^I + \sum_{n \neq 0}^{\infty} \tilde{\alpha}_{-n}^I, & \text{for the diagonal component,} \\ \exp[2iK^I \cdot \tilde{X}^I(z=1)] C(K^I), & \text{for the off-diagonal component with a root vector } K^I, \end{cases} \tag{12}$$

where $C(K^I)$ is the operator one-cocycle[2].

The amplitude (6) can be rewritten, by use of the cancelled propagator argument for the propagators $i=1$–5, as

$$\int d^{10}p \int \prod_{i=0}^{5} \frac{d^2 z_i}{\pi |z_i|^2} \mathrm{Tr} \left[\Gamma_{11} F_0 z_0^{L_0} z_0^{\tilde{L}_0^*} V_{R_2}(k_0, 1) \prod_{i=0}^{5} z_i^{L_0} z_i^{\tilde{L}_0^*} V_{R_1}(k_i, 1) \right], \tag{13}$$

where

$$V_{R_1}(k_i,z) = \{ F_0, V_{R_2}(k_0,z) \} \sim \eta_\mu^a (P^\mu + \tfrac{1}{2} \Gamma^\mu \tfrac{1}{2} k \cdot \Gamma) T^a V_0(k,z) + \eta_{\mu\nu} (P^\mu + \tfrac{1}{2} \Gamma^\mu \tfrac{1}{2} k \cdot \Gamma) \tilde{P}^\nu(z^*) V_0(k,z). \tag{14}$$

The chiral current is defined as a coefficient of polarization vector $\eta_\mu^a(k_0)$ in the effective action:

$$j_{11}^{\mu a}(k_0) = \int d^{10}p \int \prod_{i=0}^{5} \frac{d^2 z_i}{\pi |z_i|^2} \mathrm{Tr} \left[\Gamma_{11} F_0 z_0^{L_0} z_0^{\tilde{L}_0^*} V T^a \Gamma^\mu V_0(k_0, 1) \prod_{i=1}^{5} z_i^{L_0} z_i^{\tilde{L}_0^*} V_{R_1}(k_i, 1) \right]. \tag{15}$$

The divergence of this current $k_\mu \cdot j_{11}^{\mu a}$ can be evaluated, by use of a relation

$$k_\mu \Gamma^\mu V_0(k, 1) = 2\sqrt{2} i [F_0, V_0(k, 1)], \tag{16}$$

as

$$k_{0\mu}j_{11}^{\mu a} \sim \int d^{10}p \int \prod_{i=0}^{5} \frac{d^2 z_i}{\pi |z_i|^2} \left\{ \mathrm{Tr}\left[\Gamma_{11} z_0^{L_0} z_0^{\tilde{L}_0 *} T^a L_0 V_0(k_0,1) \prod_{i=1}^{5} z_i^{L_0} z_i^{\tilde{L}_0 *} V_{R_1}(k_i,1)\right] - \mathrm{Tr}\left[\Gamma_{11} z_0^{L_0} z_0^{\tilde{L}_0 *} T^a F_0 V_0(k_0,1) F_0 \prod_{i=1}^{5} z_i^{L_0} z_i^{\tilde{L}_0 *} V_{R_1}(k_i,1)\right]\right\}. \tag{17}$$

We now adopt a regularization that the integration over z_0 is responsible for all the singularity of the function. After integration over variables other than z_0, we have a singular behavior in the z_0 integration. This prescription prevents us from utilizing the cancelled-propagator argument in the z_0 integration. Therefore, the first term of (17) does not vanish. The second term of (17) turns out to be equal to the first term when we shift F_0 by using cancelled-propagator argument for $i = 1–5$.

Introducing new variables

$$\rho^i = e^{2\pi i \nu_i} = z_1 \cdots z_i, \quad \omega = e^{2\pi i \tau} = z_1 \cdots z_5 z_0 = \rho_6, \tag{18}$$

and using a relation

$$\rho^{L_0} \rho^{\tilde{L}_0 *} V(k,1) = V(k,\rho) \rho^{L_0} \rho^{\tilde{L}_0 *}, \tag{19}$$

we obtain

$$k_{0\mu}j_{11}^{\mu a} \sim \int d^{10}p \int_P d^2\tau \int \prod_{i=1}^{5} d^2\nu_i \mathrm{Tr}\left[\Gamma_{11} L_0 \omega^{L_0} \omega^{\tilde{L}_0 *} T^a V_0(k_0,1) \prod_{i=1}^{5} V_{R_1}(k_i,\rho_i)\right]. \tag{20}$$

Because of an identity $L_0 \omega^{L_0} = \omega\, d\omega^{L_0}/d\omega$, we have

$$k_{0\mu}j_{11}^{\mu a} \sim \int d^{10}p \int_{\partial P} d\tau^* \int \prod_{i=1}^{5} d^2\nu_i \mathrm{Tr}\left[\Gamma_{11} \omega^{L_0} \omega^{\tilde{L}_0 *} T^a V_0(k_0,1) \prod_{i=1}^{5} V_{R_1}(k_i,\rho_i)\right]. \tag{21}$$

The corresponding expression for the gravitational anomaly is obtained in a form similar to that of the gauge anomaly. The energy-momentum tensor is defined as a coefficient of the polarization tensor $\eta_{\mu\nu}(k_0)$ in (13):

$$T_{\mu\nu}(k_0) = \tfrac{1}{2} \int d^{10}p \int \prod_{i=0}^{5} \frac{d^2 z_i}{\pi |z_i|^2} \mathrm{Tr}\left[\Gamma_{11} F_0 z_0^{L_0} z_0^{L_0 *} V T^a (\Gamma^\mu \tilde{P}^\nu + \Gamma^\nu \tilde{P}^\mu) V_0(k_0,1) \prod_{i=1}^{5} z_i^{L_0} z_i^{\tilde{L}_0 *} V_{R_1}(k_i,1)\right]. \tag{22}$$

Then $k_\mu T^{\mu\nu}$ is evaluated by the relation (16) and

$$k_\mu \tilde{P}^\mu V_0(k,1) = \tfrac{1}{4} k^2 V_0(k,1) + 2[\tilde{L}_0, V_0(k,1)] \tag{23}$$

as

$$k_{0\mu} T^{\mu\nu}(k_0) \sim \int d^{10}p \int_{\partial P} d\tau^* \int \prod_{i=1}^{5} d^2\nu_i \mathrm{Tr}\left[\Gamma_{11} \omega^{L_0} \omega^{\tilde{L}_0 *} \tilde{P}^\nu V_0(k_0,1) \prod_{i=1}^{5} V_{R_1}(k_i,\rho_i)\right]. \tag{24}$$

Note that (21) and (24) are expressed as surface integrals over the Schwinger parameter. This situation is common to point-particle theories, in which the Schwinger parameter is one dimensional and the surface integral reduces to a point. It was already shown that the anomalies (anomalous vacuum expectation values) can be expressed as a short-time limit of the proper time (Schwinger parameter).[6]

Our procedure of obtaining anomalies is also justified by its application in the derivation of the triangle anomaly of local field theories and the hexagon gauge anomaly in type-I superstring theory[7]: The results so acquired are in agreement with the ones obtained by the Pauli-Villars regularization.[8] Therefore, our procedure can be regarded as a natural extension in string theories of the conventional regularization.

Now we evaluate the expressions (21) and (24) explicitly. Note that the gauge anomaly coming from the diagonal component of T^a resembles the gravitational anomaly. We first evaluate the gauge anomaly coming from the off-diagonal component of the external gauge fields. It is not difficult to derive the following form of anomaly in the case of the off-diagonal component of the external gauge fields[8]:

$$k_\mu j_{11}^{\mu a} \sim \epsilon(k,\eta) C(K_0) \cdots C(K_5) \int_{\partial P} d\tau^* \int \prod_{i=1}^{5} d^2\nu_i \left(\frac{-4\pi}{\ln|\omega|}\right)^5 \prod_{0 \le i < j \le 5} [\chi(\rho^j/\rho^i, \omega)]^{(1/2)(k_i \cdot k_j)} \times (\omega^*)^{-1} f(\omega^*)^{-24} \mathcal{L} \prod_{0 \le i < j \le 5} [\psi(\rho^{j*}/\rho^{i*}, \omega^*)]^{(K_i \cdot K_j)}, \tag{25}$$

where the functions in (25) have already appeared in the four-point amplitude in the new formalism,[2] and

$$\epsilon(k,\eta)=\epsilon_{\mu_1\cdots\mu_5\nu_1\cdots\nu_5}k^{\mu_1}\cdots k^{\mu_5}\eta^{\nu_1}\cdots\eta^{\nu_5}.$$

We will evaluate the behavior of (25) under the modular transformation. This transformation is generated by

$$\tau\to\tau+1,\quad \tau\to -1/\tau. \tag{26}$$

The invariance under the first transformation is easily learned. We will investigate the invariance under the second transformation $\tau\to\tau'=-1/\tau$, with $\nu_i\to\nu_i'=\nu_i/\tau$. The integral measure transforms as

$$\int d\tau^*\int\prod_{i=1}^{5}d^2\nu_i=\int d\tau'^*\int\prod_{i=1}^{5}d^2\nu_i'\frac{1}{|\tau'|^5}\frac{1}{\tau'^{*2}}, \tag{27}$$

and one can derive

$$\mathcal{L}\prod_{0\leq i<j\leq 5}[\psi(\rho^{j*}/\rho^{i*},\omega^*)]^{(K_i\cdot K_j)}=\tau'^{*8}\exp\left[\left(\frac{1}{2}\sum_{i=0}^{5}k_i^2\right)\ln(-\tau'^*)\right]\mathcal{L}'\prod_{0\leq i<j\leq 5}[\psi(\rho^{j*}/\rho^{i*},\omega'^*)]^{(K_i\cdot K_j)} \tag{28}$$

for the gauge group with self-dual root lattices, namely, the spin$(32)/Z_2$ and $E_8\otimes E_8$ groups. On the other hand, we have

$$\left(\frac{-4\pi}{\ln|\omega|}\right)^5\prod_{0\leq i<j\leq 5}[\chi(\rho^j/\rho^i,\omega)]^{(1/2)(k_i\cdot k_j)}=|\tau'|^{10}\left(\frac{-4\pi}{\ln|\omega'|}\right)^5\prod_{0\leq i<j\leq 5}[\chi(\rho^j/\rho^i,\omega')]^{(1/2)(k_i\cdot k_j)} \tag{29}$$

and the transformation property of $f(\omega^*)$ is well known;

$$(\omega^*)^{-1}f(\omega^*)^{-24}=(\tau'^*)^{-12}(\omega'^*)^{-1}f(\omega'^*)^{-24}. \tag{30}$$

The result is that (25) is invariant under the modular transformation.

Next we consider the anomaly other than the case of (25). The functions and the methods required for the evaluation have already appeared in an earlier attempt to derive the anomalies by use of the Pauli-Villars regularization.[9] The result is that the quantities $\tilde{P}^\mu$, $\tilde{P}^I$, and $\exp(2iK^I\cdot\tilde{X}^I)$ produce the factor (τ^*) under $\tau\to -1/\tau$. In the evaluation of anomalies, these equally produce the factor $(\tau^*)^6$, which plays the same role as

$$\exp\left[\left(\frac{1}{2}\sum_{i=0}^{5}K_i^2\right)\ln(-\tau'^*)\right]$$

in (28). Therefore, the expressions of anomalies (21) and (24) are invariant under the modular transformation. Evaluation of the behavior of $\tilde{P}^\mu$, $\tilde{P}^I$, and $\exp(2iK^I\cdot\tilde{X}^I)$ under the modular transformation is simplified by the method of Clavelli and Shapiro,[10] which will be reported elsewhere. As has been stated, the modular invariance of (21) and (24) leads to the absence of gauge and gravitational anomaly in heterotic-string theory.

The mechanisms of finiteness and the cancellation of gauge anomaly were shown to be identical in type-I superstring theory.[11] In heterotic-string theory, the origins of these two are attributed to one important property of closed strings: This is due to the modular invariance of amplitudes and anomalies.

[(a)]On leave of absence from Nagoya University, Nagoya 305, Japan.

[1]D. J. Gross, J. A. Harvey, E. Martinec, and R. Rohm, Nucl. Phys. **B256**, 253 (1986).

[2]D. J. Gross, J. A. Harvey, E. Martinec, and R. Rohm, Nucl. Phys. **B267**, 75 (1986).

[3]S. Yahikozawa, to be published.

[4]J. A. Shapiro, Phys. Rev. D 5, 1949 (1972).

[5]M. G. Green and J. H. Schwarz, Phys. Lett. **109B**, 444 (1982).

[6]H. Suzuki, Phys. Rev. D 33, 2948 (1986).

[7]M. B. Green and J. H. Schwarz, Nucl. Phys. **B255**, 93 (1985).

[8]H. Suzuki, National Laboratory for High Energy Physics (KEK) Report No. KEK-TH 126, 1986 (to be published).

[9]A. Sugamoto and H. Suzuki, National Laboratory for High Energy Physics (KEK) Report No. KEK-TH 125, 1986 (to be published).

[10]L. Clavelli, Phys. Rev. D 33, 1198 (1986); see also L. Clavelli and J. Shapiro, Nucl. Phys. **B57**, 490 (1973); L. Clavelli, R. H. Cox, and B. Harms, to be published.

[11]M. B. Green and J. H. Schwarz, Phys. Lett. **151B**, 21 (1985).