Motion of Extreme Reissner-Nordstrom Black Holes in the Low-Velocity Limit

G. W. Gibbons and P. J. Ruback

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 9EW, United Kingdom

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We show that in the low-velocity limit extreme Reissner-Nordstrom black holes describe approximate geodesic motion on a $3N$-dimensional parameter space. We discuss the metric and the motions in the limits that one hole is much smaller than the others and also where the holes are well separated. We also consider the quantum mechanics of these solutions.

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Despite its great interest, comparatively little is known about the motion of black holes. There are some results on the static forces between black holes indicating that only if the masses M_n and Q_n are related by the extremity condition

$$M_n = Q_n/\kappa, \tag{1}$$

where $\kappa^2 = 4\pi G$, is equilibrium possible.[1] The resulting multi-black-hole solutions are the Papapetrou-Majumdar metrics[2] whose geometry was discussed by Hartle and Hawking.[3] The metric is (for N black holes)

$$ds^2 = U^{-2}dt^2 - U^2 d\mathbf{x}^2, \tag{2}$$

where

$$U = 1 + \sum_{n=1}^{n=N} \frac{GM_n}{|\mathbf{x}-\mathbf{x}_n|}. \tag{3}$$

The solutions (for fixed masses M_n) are specified by a point $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$ in the parameter or moduli space Σ_N, which in this case is $(R^3)^N$. It is also known that if a charged black hole, whether or not it is extreme, is subjected to an external Melvin-type electric field[4] then the hole will undergo uniform acceleration.[5] In addition, D'Eath has studied the approximate motion of neutral black holes using a matched asymptotic expansion.[6]

Macroscopic charged black holes probably do not exist in nature. However, in grand unified theories (GUT's), one finds magnetic monopoles with typical masses $\sim 4\pi g^{-2} M_X$ and typical radii $\sim M_X^{-1}$. The ratio of radius to Schwarzschild radius is thus $\sim (M_X K)^{-2} g^2$. For large enough $M_X K$ and small enough g, gravitational corrections will be large and, provided $4\pi g^{-2} M_X > 2\pi g^{-1} K^{-1}$, GUT monopoles could have undergone gravitational collapse to black-hole monopoles described by the Reissner-Nordstrom geometry. By duality invariance these will interact with themselves exactly as the electrically charged holes we shall discuss in this paper.

In this Letter we wish to apply a technique which has recently been applied to a number of soliton systems in the low-velocity limit.[7-9] The basic idea is that if the motion is slow there will be negligible radiation so the system will move quasistatically through a sequence of static configurations: That is, it pursues a path in the moduli space. Radiation reaction forces scale as odd powers of a characteristic velocity, v^3 in the case of electromagnetism and v^5 for gravity. In the low-velocity limit these may be neglected in comparison with the velocity-dependent forces due to retarded interactions which scale as v^2. This is characteristic of geodesic motion. Thus in our case we seek a metric G on $(R^3)^N$ whose geodesics give the low-velocity motion of extreme Reissner-Nordstrom black holes.

An alternative viewpoint is to regard Einstein-Maxwell theory as an infinite-dimensional dynamical system[10] with total energy given by the Arnowitt-Deser-Misner[11] mass M. The static solutions (2) correspond to a $3N$-dimensional submanifold Σ_N of the configuration space in which the potential energy functional is constant. Now consider all configurations which are asymptotically flat and regular outside N apparent horizons carrying charge Q_n. Earlier results[12,13] show that the total energy M attains its lower bound on the submanifold Σ_N consisting of static configuration given by (2) and (3). If this system is given a small amount ($\ll Mc^2$) of kinetic energy, its subsequent motion must be confined to a small neighborhood of Σ_N by energy conservation (at least in the classical case when quantum tunneling is not allowed). Since the potential energy is constant along Σ_N, the effective Lagrangean will be given by just the kinetic-energy terms, which is to say that approximately it will execute geodesic motion along Σ_N. This approximation will get better and better as the characteristic velocity gets smaller and smaller.

A variety of methods have been used to evaluate metrics on moduli spaces. In the present case they would correspond to the following.

(1) One could consider solutions of the linearized Einstein-Maxwell equations about a particular solution p on Σ_N which are independent of time ("zero modes"). These zero modes span the tangent space $T_p(\Sigma_N)$ to Σ_N at p. Substitution of the perturbation

into the Hamiltonian retaining only quadratic terms gives the metric G. This has been done for Kaluza-Klein monopoles.[7]

(2) One might substitute into the Hamiltonian an *Ansatz* in which the parameters of the solution are allowed to be time dependent and retain only terms which are second order in the generalized velocities. This has been done for CP^1 solitons.[8]

(3) General considerations might give special knowledge about G which would permit its evaluation. This was possible in the case of Yang-Mills monopoles.[9]

At present none of these techniques appears to be tractable in our case. However, some information may be gained about the metric G in certain asymptotic limits.

(1) One can consider the limit on which one of the black holes is very much less massive than N others. It is physically plausible that its motion will be given by a four-dimensional world line in the metric (2) satisfying the point-particle equations of motion with mass m and charge q related by $m=q/\kappa$. Using the Hamilton-Jacobi method one gets the equations

$$m\, dx/d\lambda = U^{-2}\nabla W, \tag{4}$$

$$m\frac{dt}{d\lambda} = U^2\frac{E}{c^2}+\frac{q}{\kappa}U, \tag{5}$$

where λ is proper time along the world line and E is the conserved energy, and where W satisfies (by virtue of the relation $m=q/\kappa$)

$$|\nabla W|^2 = U^4\frac{E^2}{c^2}+2mEU^3. \tag{6}$$

A particle at rest has $E=0$ and one moving slowly has $E \ll mc^2$. If U is not too large (i.e., the particle stays away from the horizons), Eqs. (4)–(6) reduce to

$$m\, dx/dt = U^{-3}\nabla W, \tag{7}$$

where

$$U^{-3}|\nabla W|^2 = 2mE. \tag{8}$$

Equations (7) and (8) are precisely the Hamilton-Jacobi equations for nonrelativistic geodesic motion on the three-manifold $\Pi \equiv R^3 - \{\mathbf{x}_n, n=1,\dots,N\}$

$$H_{ij}dx^i dx^j = mU^3 d\mathbf{x}^2. \tag{9}$$

One should think of Π, in the limit as $m\to 0$, as a totally geodesic submanifold of the $(N+1)$-particle moduli space Σ_{N+1}. The equation of motion coming from (7) and (8) is

$$\frac{d(U^3\dot{\mathbf{x}})}{dt} = \tfrac{3}{2}\dot{\mathbf{x}}^2U^2\nabla U. \tag{10}$$

Equation (10) indicates that two extreme black holes experience, at large distances, a Newtonian force F (define by $m\ddot{x}$) given by

$$F=(3GM_1M_2/2c^2r^2)[2\mathbf{v}(\mathbf{v}\cdot\hat{\mathbf{r}})-\mathbf{v}^2\hat{\mathbf{r}}], \tag{11}$$

where $\mathbf{v}$ is their relative velocity. Equation (11) shows that two extreme black holes moving radially experience a repulsive force at large distances. The general, nonradial, motion is described below.

(2) Another way of obtaining information about the behavior of the metric G at large separations is to imitate Manton's calculation in the Yang-Mills case by use of Lienard-Wiechert potentials.[14]

The equations of motion for gravitating bodies has received much study. Einstein, Infeld, and Hoffman[15] showed that one could obtain equations of motion in terms of the asymptotic field, irrespective of its source. In the lowest post-Newtonian approximation in which radiation reaction forces and terms of higher order in v^4/c^4 are neglected, one obtains a set of equations for the positions of the bodies depending on their masses but no high multipole moments (see also Eddington and Clark[16]). The equations may be derived from the Lorentz-Droste-Fichtenholz Lagrangean.[17,18] An outline derivation is given by Landau and Lifschitz[19] (see their Eq. 106.17). A critical and historical review has been given by Damour.[20] To include the effect of electromagnetic forces we add to the Lorentz-Droste-Fichtenholz Lagrangean Darwin's terms due to the electric charges (Ref. 19, Eq. 65.7) to get for general M_n and Q_n

$$L=\sum_{n=1}^{n=N}\tfrac{1}{2}M_n\mathbf{v}_n^2\left(1+\frac{1}{4}\frac{\mathbf{v}^2n}{c^2}+3\frac{G}{c^2}\sum_{m\neq n}\frac{M_m}{r_{mn}}\right)+\sum_{n>m}\frac{1}{r_{mn}}\left(GM_nM_m-\frac{Q_nQ_m}{4\pi}\right)$$
$$+\sum_{n>m}\frac{1}{2c^2}\frac{1}{r_{mn}}\left[\mathbf{v}_n\cdot\mathbf{v}_m\left(\frac{Q_mQ_n}{4\pi}-7GM_nM_m\right)+(\mathbf{v}_n\cdot\hat{\mathbf{r}}_{mn})(\mathbf{v}_m\cdot\hat{\mathbf{r}}_{mn})\left(\frac{Q_mQ_n}{4\pi}-GM_nM_m\right)\right], \tag{12}$$

where $\mathbf{r}_{mn}$ is the distance between the mth and nth particles whose velocities are $\mathbf{v}_m$ and $\mathbf{v}_n$, respectively, and $\hat{\mathbf{r}}_{mn}$ is a unit vector in the direction joining m to n. If the Bogomolnyi condition (1) holds, the static force vanishes. In the slow-motion limit we get $c\to\infty$ but with GM_n/c^2 tending to a nonzero limit (i.e., slow motion but strong gravity). This gives the metric

$$\sum_{n=1}^{n=N}M_nd\mathbf{x}_n^2+\sum_{m<n}\frac{3G}{c^2}\frac{M_nM_m}{r_{mn}}(d\mathbf{x}_m-d\mathbf{x}_n)^2. \tag{13}$$

If we restrict all the $d\mathbf{x}_n$'s in (13) except 1 to vanish (i.e., we look at the metric induced on the submanifold Π) the metric (13) agrees (up to a trivial factor) with (9) at large distances. This confirms that the black holes experience a velocity-dependent repulsion at large distances.

It is interesting to note that metric (13) is asymptotically flat in those directions in which *all* $r_{mn} \rightarrow \infty$ and that it is Ricci flat in that limit. However, metric (9) is not Ricci flat everywhere, which strongly suggests that the exact metric on Σ_N is neither Ricci flat nor has a vanishing Ricci scalar.

The metric (9) is complete and the holes could only touch after an infinite time. *However,* the slow-motion approximation will break down long before that happens. Some idea of the reliability of the slow-velocity approximation can be obtained by comparison of the motion of a small particle with $m = q/\kappa$ given by Eqs. (4) and (5) with that obtained from slow-velocity Eqs. (7) and (8). For just one big black hole we have central-force motion. We find that an infalling small black hole with energy $E < sJ^2c^4/27mG^2M^2$, where J is the angular momentum, will [according to Eqs. (7) and (8)] reach a minimum radius, which is always greater than a critical radius equal to $GM/2c^2$, and bounces out again. If $E > 2J^2c^4/27G^2M^2m$ it has no turning point. According to the more accurate Eqs. (4)–(6) this behavior is quantitatively correct but the critical energy is

$$E = \frac{J^2c^2}{G_m M} r_0 \frac{GM/c^2 - r_0}{r_0 + Gm/c^2)^3}, \tag{14}$$

where the critical radius now depends on J^2 through the equation (which can easily be solved iteratively)

$$r_0 = \frac{GM}{2c^2} + \frac{J^2}{2m^2GM}\left(\frac{Gm - r_0c^2}{Gm + r_0c^2}\right)^2. \tag{15}$$

Equations (14) and (15) agree with the nonrelativistic results given above in the limit $\epsilon = J^2c^2/m^2(GM)^2 \rightarrow 0$. For nonvanishing values of ϵ they give a larger value for the critical radius.

If the black holes move so slowly that their de Broglie wavelengths are comparable with the horizon size GM/c^2 one should treat the motion quantum mechanically. This may be done by the writing of a covariant Schrödinger equation on the moduli space Σ_N by use of the metric G. Such a Schrödinger equation might contain a term proportional to the Ricci scalar of G, which could arise from integrating over classical paths on Σ_N and a potential term arising from quantum fluctuations orthogonal to Σ_N. Some idea of the expected quantum-mechanical behavior can be obtained by neglect of these latter terms and equating the Hamiltonian to the covariant Laplacian on Σ_N (cf. the Yang-Mills case[21]).

The quantum-mechanical treatment becomes especially interesting if the masses of all the black holes are equal. This could happen, for instance, if one considered a theory in which charge was quantized. It would also happen if one considered magnetically charged black holes whose magnetic-monopole charge obeyed the Dirac quantization condition. In either case the classical black holes would be indistinguishable and the moduli space would now be (if we did not allow the black holes to coalesce) $\tilde{\Sigma}_N = [(R^3)^N - \Delta_N]/S_N$ where Δ_n consists of those configurations for which $r_{mn} = 0$ for some $m \neq n$ and S_N is the permutation group on N objects. The fundamental group $\pi_1(\tilde{\Sigma}_N)$ is now S_N. There are in principle two possible quantizations, one in which S_N is represented on the wave functions by $+1$ (i.e., the black holes are bosons) or more speculatively that in which odd permutations in S_N are represented by -1 and even permutations act trivially (i.e., the black holes behave as fermions).

If one black hole is very much smaller than N others the corresponding Schrödinger equation will be the nonrelativistic limit of the charged Klein-Gordon equation on the Papapetrou-Majumdar background produced by the N large black holes. To get some idea of what is involved consider the $N = 1$ case. Solutions of the Klein-Gordon equation have the form

$$\Psi = \frac{1}{r} Y_{1m}(\theta, \phi) \exp(-iEt) R(r), \tag{16}$$

where

$$\frac{d^2R}{dr^2} + \left\{\left(1 + \frac{GM}{c^2r}\right)^4 E^2 + 2E_m\left(1 + \frac{GM}{c^2r}\right)^3 - l\frac{(l+1)}{r^2}\right\}R = 0. \tag{17}$$

The quantum behavior is similar to the classical behavior except that now quantum tunneling is allowed so that even small holes with energy less than the critical energy can fall into the big hole. If there are two large holes the tunneling will be more complicated and the possibility exists for tunneling from the region near one horizon to that near another in a manner which is reminiscent of tunneling in molecules.

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[1]H. Müller zum Hagen, D. O. Robinson, and H. J. Seifert,

Gen. Relativ. Gravitation **5**, 61 (1974); G. W. Gibbons, Commun. Math. Phys. **35**, 13 (1974), and Proc. Roy. Soc. London, Ser. A **372**, 535 (1980).

[2]A. Papapetrou, Proc. Roy. Irish Acad., Sect. A **51**, 191 (1947); S. D. Majumdar, Phys. Rev. **72**, 39 (1947).

[3]J. B. Hartle and S. W. Hawking, Commun. Math. Phys. **26**, 87 (1972).

[4]M. A. Melvin, Phys. Lett. **8**, 65 (1964).

[5]F. J. Ernst, J. Math. Phys. **17**, 515 (1976); J. Bicak, Proc. Roy. Soc. London, Ser. A **371**, 429 (1980).

[6]P. D. D'Eath, Phys. Rev. D **12**, 2183 (1975).

[7]P. J. Ruback, University of Cambridge, Department of Applied Mathematics and Theoretical Physics Report, 1986 (to be published).

[8]R. Ward, Phys. Lett. **158B**, 424 (1985); A. M. Din and W. J. Zakrzewski, Nucl. Phys. **B259**, 667 (1985).

[9]M. F. Atiyah and N. J. Hitchin, Phys. Lett. **107A**, 21 (1985); M. F. Atiyah and N. M. Hitchin, Philos. Trans. Roy. Soc. London, Ser. A **315**, 459 (1985).

[10]A. E. Fischer and J. E. Marsden, in *General Relativity: An Einstein Centenary Survey,* edited by S. W. Hawking and W. Israel (Cambridge Univ. Press, Cambridge, 1979).

[11]R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. **116**, 1322 (1959).

[12]G. W. Gibbons and C. M. Hull, Phys. Lett. **109B**, 190 (1982).

[13]G. W. Gibbons, S. W. Hawking, G. T. Horowitz, and M. J. Perry, Commun. Math. Phys. **88**, 295 (1983).

[14]N. S. Manton, Phys. Lett. **154B**, 397 (1985), and Phys. Lett. **157B**, 475 (E) (1985).

[15]A. Einstein, L. Infeld, and B. Hoffman, Ann. Math. **39**, 65 (1938).

[16]A. Eddington and G. L. Clark, Proc. Roy. Soc. London, Ser. A **166**, 465 (1938).

[17]H. A. Lorentz and J. Droste, Versl. K. Akad. Wet. Amsterdam **26**, 392, 649 (1917), reprinted in *The Collected Papers of H. A. Lorentz* (Nijhoff, The Netherlands, 1937), Vol. 5, p. 330.

[18]I. G. Fichtenholz, Zh. Eksp. Teor. Fiz. **20**, 824 (1950).

[19]L. D. Landau and E. M. Lifschitz, *Classical Theory of Fields, 4th Ed.* (Pergamon, Oxford, 1975).

[20]T. Damour, in *Gravitation Radiation,* edited by N. Deruelle and Tsvi Piran (North-Holland, Amsterdam, 1983).

[21]G. W. Gibbons and N. S. Manton, Nucl. Phys. B (to be published).

[22]D. Maison, in *Supersymmetry and Supergravity,* edited by B. Milewski (World Scientific, Singapore, 1983).

[23]W. Israel and G. A. Wilson, J. Math. Phys. **13**, 865 (1972).