Spatial and Temporal Patterns in High-Power Ferromagnetic Resonance

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Previous work on bifurcations and chaos in ferromagnetic resonance has singled out only the most unstable pair of spin waves for consideration, even though in general an entire manifold of spin wave is elegible for instability. This serious defect is removed here, and, as the result, we are able to calculate analytically the frequency of the limit cycle and its accompanying distinctive spatial correlation patterns.

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The recent revival of experiments on,[1-3] and theory of,[1-6] high-power ferromagnetic resonance has been phrased in terms of some of the modern concepts of nonlinear dynamics.[7] So far the theoretical work has been largely numerical and has concentrated on the fate of a single spin-wave pair as the rf excitation is increased beyond a certain threshold.[8] Well beyond that threshold, limit cycles, chaos, windows, etc. were observed both in the laboratory[1] and also in computations. The latter have concentrated on an initially unstable spin-wave pair only.

However, as the signal increases beyond threshold, an entire manifold of pairs enters the picture. In fact, for parallel pumping as well as for the subsidiary resonance, this is already the case at threshold.[8,9]

We begin by re-emphasizing the old observation of Ref. 8 that nothing spectacular happens at, and somewhat above, threshold. This is easy to see for perpendicular pumping of the main resonance. As the uniform precession angle increases towards threshold, the thermally excited spin waves are less attenuated and reach a higher level. This higher level feeds back into the motion of the uniform mode as additional loss, tending to reduce the precession angle. At threshold the spin-wave level is very high but finite, and the precession angle tends to stick at its threshold value. This situation continues even beyond threshold. Recently, however, we found that eventually this steady state becomes unstable, undergoing a Hopf bifurcation. At yet higher power levels, further bifurcations and eventually chaotic states occur. We show here that the first Hopf bifurcation (which results in a limit cycle) sets up a collective mode of the entire manifold of spin waves degenerate with the uniform mode.

In Ref. 8, the steady state was considered classically, at a finite temperature, and the spin-wave level, though abnormally large, was still proportional to the mean square thermal fluctuation field. Actually it is formally simpler to go to zero temperature, allowing the zero-point motion to replace the thermal agitation.

Consider the Hamiltonian reduced to the uniform mode and the degenerate spin-wave manifold:

$$H=\omega_0 a_0^\dagger a_0+\sum\omega_k b_k^\dagger b_k+\sum\rho_k b_k^\dagger b_{-k}^\dagger a_0 a_0+\omega_s\exp(i\omega t)a_0^\dagger+\text{c.c.},\tag{1}$$

where a_0 and the b_k are respectively the uniform-precession and spin-wave amplitudes. ω_s and ω are the signal amplitude and frequency. ω_0 and ω_k are the natural frequencies of the uniform mode and of the spin waves, these two being equal in the case of the main resonance instability. The equations of motion are

$$i\dot{B}_k=-\rho_k B_{-k}^\dagger A_0^2-i\eta_k B_k,\tag{2a}$$

$$i\dot{A}_0=\omega_s-\sum\rho_k^* B_k B_{-k} A_0^\dagger-i\eta_0 A_0,\tag{2b}$$

$$B_k=b_k\exp(-i\omega_k t),\quad A_0=a_0\exp(-i\omega_0 t),$$

where the η's denote intrinsic damping constants. The B's are subjected to the Bogoliubov transformation[10]

$$B_k=u_k\beta_k-v_k\beta_{-k}^\dagger,\quad B_{-k}^\dagger=u_k^*\beta_{-k}^\dagger-v_k^*\beta_k,$$

where

$$u_k=\cosh(\psi_k/2),\quad v_k=\sinh(\psi_k/2)\exp(2i\phi_k),$$

$$\cosh(\psi_k)=i\eta_k/\Omega_k,$$

$$\sinh(\psi_k)\exp(2i\phi_k)=\rho_k A_0^2/\Omega_k,$$

and

$$\Omega_k=i(\eta_k-\rho_k|A_0|^2).$$

In the vacuum state of the β's, the expectation values of B_kB_{-k} and $B_k^\dagger B_k$ are

$$\langle B_kB_{-k}\rangle=-\rho_kA_0^2/2\Omega_k,\tag{3a}$$

$$\langle B_k^\dagger B_k\rangle=-\tfrac{1}{2}+\tfrac{1}{2}(1-\rho_k|A_0^2|/\eta_k)^{-1}.\tag{3b}$$

The last result shows that the magnon occupation number tends to peak sharply for the mode or modes for which $\rho_k|A_0|^2$ is close to η_k. But it does not become infinite, as is seen by substitution of it in (2b), which becomes an equation for the steady-state value of $|A_0|$:

$$|A_0| = \frac{\omega_s}{\eta_0 + \sum|\rho_k|^2|A_0|^2/2(\eta_k - \rho_k|A_0|^2)}. \quad (4)$$

(As a check, the same procedure was used for the subsidiary resonance, and the result of Ref. 8 was obtained, with the measure of zero-point fluctuation replacing the thermal field used there.) As ω_s is increased beyond threshold, $|A_0|$ is seen to stick near the smallest value of ρ_k/η_k. This and the corresponding distribution of magnon numbers (3a) is the only stable fixed point of the system.

We now find that as ω_s is further increased to a new threshold ω_s^1, a Hopf bifurcation occurs and a limit cycle is formed. That threshold is found by examining the equations of motion of $\delta B_k = B_k - \bar{B}_k$ and of $\delta A_0 = A_0 - \bar{A}_0$, where the barred quantities are the values at the above fixed point. A collective mode is found with complex eigenvalue whose real part goes to zero at ω_s^1. The linearized equations for the increments are

$$\delta\dot{B}_k = i\rho_k A_0^2 \delta B_{-k}^* + 2i\rho_k A_0 B_{-k}^* \delta A_0 - \eta_k \delta B_k, \quad (5a)$$

$$\delta\dot{A}_0 = i\sum \rho_k B_k A_0^* \delta B_{-k} + i\sum \rho_k B_{-k} A_0^* \delta B_k + i\sum \rho_k B_k B_{-k} \delta A_0^* - \eta_0 \delta A_0. \quad (5b)$$

With the assumption that all increments vary like $\exp(\lambda t)$ with time, it is easily found that λ is given by the secular equation

$$(\lambda+\eta_0)\left\{1+2|A_0|^2\sum\rho_k^2\left[1+\left(1-\frac{\rho_k|A_0|^2}{\eta_k}\right)^{-1}\right][(\lambda+\eta_k)^2-\rho_k^2|A_0|^4]^{-1}\right\}$$
$$= \frac{|A_0|^2}{2}\sum\frac{\rho_k^2}{(\eta_k-\rho_k|A_0|^2)}\{1+4\rho_k^2|A_0|^4[(\lambda+\eta_k)^2-\rho_k^2|A_0|^4]^{-1}\}. \quad (6)$$

It is seen that the spectrum of λ consists of a band of "single particle" decay constants, very nearly equal to $-\eta_k + \rho_k|A_0|$, and an isolated pair of complex conjugate roots whose equal real parts pass through zero from below as ω_s is increased through ω_s^1. The remaining imaginary parts are $\pm i\lambda''$, where λ'' is the frequency of the resulting limit cycle. Taking real and imaginary parts of the secular equation at this bifurcation threshold, one obtains two equations for λ''. When we bear in mind that $|A_0|$ is close to $(\eta_k/\rho_k)_{\min}$, these equations simplify to (from now on the k dependence of η is ignored)

$$\lambda''^2 = 2|A_0|^2\eta\Sigma + 4\eta^2/(|A_0|^2\Sigma/2\eta - 1), \quad (7a)$$

$$\lambda''^2 = 2|A_0|^2\eta\Sigma - 4\eta^2, \quad (7b)$$

where

$$\Sigma = \sum\rho_k^2/(\eta_k - \rho_k|A_0|^2)$$

can be expressed through (4) as a function of $|A_0|$ and ω_s. These equations are satisfied by only one value of ω_s, $\omega_s/|A_0| = 4\eta$. The corresponding $\lambda'' = 2\sqrt{2}\eta$; typical damping constants of yttrium iron garnets are from 0.1 to 5 G, so that the limit-cycle frequency ranges from 200 kHz to 10 MHz.

Our theory thus furnishes a possible explanation, as well as an analytic expression, for the auto-oscillation frequency observed by several investigators[2,11,12] and hitherto calculated only by numerical simulations for a single mode pair. Also, our theory is a viable alternative to somewhat *ad hoc* arguments concerning standing waves.[12] Furthermore, the observed temperature dependence of the auto-oscillation frequency[11] is in qualitative accord with our expression, if we bear in mind the temperature dependence of η discussed by Kasuya.[13]

We first discuss the spatial pattern of excitation, below the threshold for Hopf bifurcation, which may be characterized by the correlation function of the transverse magnetization:

$$c(z) = \langle m^+(\mathbf{r}+\mathbf{z})m^-(\mathbf{r})\rangle, \quad (8)$$

where

$$m^+ = m_x + im_y,$$

and the z axis is parallel to the dc field, so that

$$c(z) = \sum|B_k|^2\exp(ikz).$$

(For simplicity we separately work out correlation along and perpendicular to the dc field.) Since the Hamiltonian we have used is restricted to the submanifold $\omega_k = \omega_0$ (although this is not necessary[14]), we only sum over $\mathbf{k}$'s meeting this restriction and multiply the integral by a width Δ over which the integral is large. Thus

$$c(z) = \Delta\eta\int k^2\,dk\,\sin\theta\,d\theta\,\exp(ikz\cos\theta)\delta(\omega_k - \omega_0)/(\eta - \rho_k|A_0|^2), \quad (9)$$

where

$$\rho_k = (\omega_{ex} l^2 k^2 - N_z \omega_m + \omega_m \cos^2\theta_k)/2.$$

$\omega_m = 4\pi M$, l is the lattice spacing, ω_{ex} is the exchange field, and N_z is demagnetizing factor. For large z, this integral becomes

$$c(z) = F(\omega_0, \omega_m, |A_0|^2)\exp[ik_0 g(k_0) z]/\sqrt{z} + O(z^{-1/2-\alpha}), \tag{10}$$

where $g(k)$ is the value of $\cos\theta$ on $\omega_k = \omega_0$. The spatial oscillation frequency is

$$k_0 g(k_0) = \left\{ \frac{1}{\omega_{ex} l^2} \left[w_0 \left(\frac{y_0}{y_0 + \omega_m} \right)^{1/2} - y_0 \right] \left\{ \frac{\omega_0}{\omega_m} \left[\left(\frac{y_0}{y_0 + \omega_m} \right)^{1/2} - \left(\frac{y_0 + \omega_m}{y_0} \right)^{1/2} \right] + 1 \right\} \right\}^{1/2}.$$

k_0 is the stationary phase point of the integrand and $y_0 = \omega_0 - N_z \omega_m$. The width Δ is chosen such that the corresponding frequency shift $\Delta\omega$ is of the same order as η:

$$\Delta = \frac{\Delta\omega}{|\nabla_k \omega|} = \frac{\eta}{2} k_0 \left\{ \left[\omega_0 \left(\frac{y_0}{y_0 + \omega_m} \right)^{1/2} - y_0 \right]^2 + \frac{1}{9} \left(\frac{\omega_m}{\omega_0} \right)^2 y_0^2 \right\}^{-1/2},$$

which simplifies to

$$\Delta = \tfrac{3}{2} \eta k_0 / \omega_m$$

for $\omega_0 >> N_z \omega_m$. The coefficient

$$F(\omega_0, \omega_m, |A_0|^2) = \frac{\Delta k_0 \eta \exp(\frac{1}{4} i\pi)}{(\eta - \rho_{k_0} |A_0|^2)\{[kg(k)]''|_{k=k_0}\}^{1/2}}$$

turns out to be an extremely lengthy expression. In the same approximation, the correlation perpendicular to the dc field is

$$c(x) = \Delta\eta \int d^3k \exp(ikx \sin\theta \cos\phi) \frac{\delta(\omega_k - \omega_0)}{\eta - \rho_k |A_0|^2} = G(\omega_0, \omega_m, |A_0|^2) \frac{\exp[ik_0 \xi(k_0) x]}{x} + O(x^{-1-\beta}),$$

where $\xi(k_0)$ is the value of $\sin\theta$ on $\omega_k = \omega_0$, and

$$k_0 \xi(k_0) = \left\{ \frac{1}{\omega_{ex} l^2} [-y_0 + (\omega_0^2 y_0)^{1/3}] \frac{1}{\omega_m} \left[\frac{\omega_0^2}{(\omega_0^2 y_0)^{1/3}} - (\omega^2 y_0)^{1/3} \right] \right\}^{1/2},$$

$$G(\omega_0, \omega_m, |A_0|^2) = \Delta k_0 \frac{\eta}{\eta - \rho_{k_0} |A_0|^2 [2/\pi k_0 \xi(k_0)]^{1/2} \{[(k\xi(k)]''|_{k=k_0}\}^{-1/2}},$$

$$[k\xi(k)]'|_{k=k_0} = 6[y_0 - (\omega^2 y_0)^{1/3}]/k_0 \xi(k_0) \omega_m, \quad k_0 = \{(1/\omega_{ex} l^2)[-y_0 + (\omega^2 y_0)^{1/3}]\}^{1/2}.$$

The correlation decays faster in the transverse than in the longitudinal direction.

At higher temperatures a finite thermal field replaces the effect of zero-point fluctuations. Then the spatial pattern becomes the convolution of the exponentially decaying correlation function of the thermally excited transverse magnetization (i.e., the Fourier transform of the magnon occupation number) and the low-temperature correlation given above. Since the thermal correlation function is of very short range, the result is nearly the same as at low temperatures. Turning now to the correlation pattern change induced by the Hopf bifurcation, we have

$$\delta c(\mathbf{r}) = \sum (\delta B_k^* B_k + \delta B_k B_k^*) \exp(i\mathbf{k}\cdot\mathbf{r}),$$

with δB_k obtained from (5) to within a multiplicative constant:

$$B_k^* \delta B_k + B_k \delta B_k^* = |B_k| |\delta B_k| \cos(\nu_k + \nu_{\lambda''} + \nu_{b_k} - \nu_a + \lambda'' t)$$

where ν_{b_k} and ν_a are the phases of b_k and a, respectively. Since they are locked to each other through (3a), we can choose $\nu_a - \nu_{b_k}$ to be zero:

$$\nu_{\lambda'} = \tan^{-1}[(\lambda''^2 + 2\eta^2)/(-\eta\lambda'')] = -\tfrac{2}{5}\pi,$$

and

$$\nu_k = \tan^{-1}\left(\frac{4|A_0|^2\eta^2\rho_k/\lambda''(\lambda''^2+4\eta^2)}{1+4|A_0|^2\eta\rho_k/(\lambda''^2+4\eta^2)}\right) = \tan^{-1}\left(\frac{1}{4\sqrt{2}}\frac{\rho_k}{3\rho_{k_{\max}}+\rho_k}\right),$$

$$\delta c(\mathbf{r}) = \frac{\Delta}{\eta}\int d^3k\,\delta(\omega_k-\omega_0)\frac{\rho_k^2|A_0|^3}{\eta-\rho_k|A_0|^2}\cos(\nu_k-\tfrac{2}{5}\pi+2\sqrt{2}\eta t)\exp(i\mathbf{k}\cdot\mathbf{r}).$$

Since ν_k has a very small range of variation, from 0° to 2.5°, one can apply the same calculation procedure as before. The final results of the change of correlation have the same form as those before the bifurcation except for an additional factor $\rho_{k_0}^2|A_0|^3\cos(\nu_{k_0}-\frac{2}{5}\pi+2\sqrt{2}\eta t)$ which shows that the correlation pattern formed after the limit-cycle bifurcation is a standing wave at least approximately.

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