Dimensionality of Strange Attractors Determined Analytically

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An analytical method to determine the dimensionality of strange attractors in two-dimensional maps is introduced. In this method, the geometric structures of an attractor are obtained from a procedure developed previously. Such structures often appear to be the Cartesian product of a curve and a Cantor set. From the geometric structures, we determine the Hausdorff dimension first for the Cantor set, and then for the attractor. The results compare well with numerical results obtained for the Hénon, Zaslavskii, and Kaplan-Yorke maps.

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Strange attractors, consisting of interweaving trajectories, are found in many dissipative chaotic systems.[1] The typical geometry of a strange attractor is a set of self-similar structures that repeat on finer and finer scales. A strange attractor has a fractal dimensionality[2] that can characterize certain phenomena among various fields in physics such as fluid dynamics,[3,4] geophysics,[5] chemical thermodynamics,[6] medical physics,[7] etc. Several methods to determine the dimensionality of strange attractors have been introduced. The primitive box-counting algorithm has been found impractical.[8] The Kaplan-Yorke conjecture,[9,10] though intuitive, encounters difficulties in measuring the Lyapunov exponent under some circumstances.[11] Most other methods[11,12] involve tedious bookkeeping of data from simulations or experiments, such as a time series or the distances between datum points. However, I believe that the geometry of the attractor itself plays a more basic role in determining the dimension. In this Letter I present the first analytical method to determine the (Hausdorff) dimension of a strange attractor, simply by examining the geometric structures of an attractor in a two-dimensional dissipative map. I note that a strange attractor of a two-dimensional map appears to be a Cantor one-manifold, i.e., the Cartesian product of a one-dimensional manifold and a Cantor-setlike structure with dimension σ between 0 and 1. If we can find σ by examining the cross section, we can conclude that the attractor is of dimension $d=1+\sigma$.

Let us here consider a Cantor set constructed as follows: Define a closed interval C_0, say [0,1], as generation zero; form set C_1 (generation one) by erasing $k-1$ open segments from the mother interval C_0 so that k daughters (closed intervals) each of length ϵ_i $(i=1,\ldots,k)$ times the mother interval remain; form set C_2 (generation two) by repeating this process with each of the k daughters, *ad infinitum.* The set thus constructed is called a rescaling Cantor set, since each daughter resembles its mother. From Farmer, Ott, and Yorke[13] it follows that the dimension σ satisfies

$$g(\sigma)=\sum_{i=1}^{k}\epsilon_i^{\sigma}=1. \tag{1}$$

We call $g(\sigma)$ the *fractal generator,* which characterizes the construction of a rescaling Cantor set.

It seems that it would be easier to determine the capacity of a Cantor set, but instead we choose the Hausdorff dimension for two reasons. First, the capacity $\sigma=\lim_{\epsilon\to 0}\log n(\epsilon)/|\log\epsilon|$ seems easy to evaluate only because $n(\epsilon)$ has often been taken to be the number of cells needed to cover the set at a certain generation, as in the familiar case of the middle-third Cantor set. However, in general, this formula is only valid if $n(\epsilon)$ represents the number of cells needed to cover the Cantor set itself. For the asymmetric Cantor set discussed by Farmer[14] as an example, the fractal generator is $(\frac{1}{4})^{\sigma}+(\frac{1}{2})^{\sigma}$. The capacity log3/log4 $=0.7925$ obtained in Ref. 14 is wrong,since it only corresponds to a fractal generator $3(\frac{1}{4})^{\sigma}$. The author in Ref. 14 had assumed that $n(4^{-i})=3^i$. In fact, $n(\frac{1}{4})=3$, but $n(\frac{1}{16})=8\neq 9$. An expression for $n(\epsilon)$ is not trivial at all![15] Equation (1), however, gives $\sigma=\log\Omega/\log(\frac{1}{2})=0.6942$. Here Ω is the golden mean. This value of σ is much closer to the information dimension 0.6887 given by Farmer.[14] We see that the Hausdorff dimension is easy to generalize to nonuniform Cantor sets (i.e., with unequal ϵ_i's).

Second, capacity is not invariant under coordinate changes.[16] Ott, Withers, and Yorke[16] suggested that the word "dimension" be reserved only for those quantities having this invariance property, such as the Hausdorff dimension.

Generally, Eq. (1) cannot be solved in closed form. The middle-third and the asymmetric Cantor sets mentioned above are only exceptional cases. However, we can use Newton's method. We choose an initial guess $\sigma^{(0)}=0$ and then iterate

$$\sigma^{(j+1)}=\sigma^{(j)}-\left(\sum_i\epsilon_i^{\sigma^{(j)}}-1\right)/\left(\sum_i\epsilon_i^{\sigma^{(j)}}\ln\epsilon_i\right).$$

A four-significant-figure accuracy usually requires no more than three or four iterations.

We have previously developed a procedure to determine analytically the equilibrium-invariant distribution on strange attractors in two-dimensional dissipative maps.[17,18] Now we can examine the structure of an attractor and compare its cross section with the Cantor set mentioned above. I simplify the procedure and describe it below.

Consider a dissipative map T that maps z to z', where $z=(x,y)$ are coordinates of the two-dimensional plane (usually the angle-action variables of the corresponding Hamiltonian system, if it exists). I first choose a simple trapping region C^0 of the map, say the smallest rectangular region such that $z\in C^0$ implies $z'=T(z)\in C^0$. To obtain the equilibrium invariant distribution, I start with a generation-zero distribution

$$f^{(0)}(z)\begin{cases} >0, & z\in C^0,\\ =0, & z\notin C^0.\end{cases}$$

Structures of a subsequent generation (n) are obtained by applying T repeatedly (n times) on C^0. We can write

$$f^{(n)}(z)\begin{cases} >0, & T^{-n}(z)\in C^0 \text{ or } z\in C^n,\\ =0, & T^{-n}(z)\notin C^0 \text{ or } z\notin C^n,\end{cases}$$

where $C^n=T^n(C^0)$. The equilibrium invariant distribution is given by $f^{(\infty)}$ and the attractor is given by C^∞.

As an illustration, consider the dissipative standard map[17–20] $[y'=by+k\sin 2\pi x,\ x'=x+y'(\mathrm{mod}1)]$ with $b=0.05$ and $k=1.5$. An appropriate trapping region for $f^{(0)}$ is $|y|\leq\bar{y}=k/(1-b)$. Successive generations $f^{(1)},\ldots,f^{(4)}$ are shown in Fig. 1. Observe that each middle stripe among the seven bears a new generation of seven daughter stripes. The rescaling property is apparent from the figure. Yet, it is a premature conclusion that the cross section is of dimension σ given by Eq. (1). To see the picture more clearly, define the set H_1 (H_n) to be the segment(s) near (0.5,0) that C^1 (C^n) intersects the x axis, and the set $V_1(\xi)$ to be the segments that C^1 intersects the vertical line $x=\xi$. Notice that H_1 has seven daughters (H_2) resembling $V_1(0.5)$. The middle sister in H_2 also has seven daughters resembling $V_1(0.5)$. However, each of the other six sisters in H_2 has five, six, or seven daughters resembling $V_1(\xi)$ with ξ (other than 0.5) satisfying $T^2(\xi,0)\in H_1$. In fact, exactly seven points on the x axis have images (under T^2) in H_1. We expect that for large n, sisters (or cousins) in H_n have daughters resembling $V_1(\xi)$ with ξ distributed compactly between 0 and 1. It is evident that the cross section cannot be described by a rescaling Cantor set. Nevertheless, I can model the cross section of an attractor by interweaving offsprings, each characterized by its own fractal generator.

Let us now consider a probabilistic Cantor set, constructed such that at generation infinity, the probability that each mother interval bears daughters as prescribed by the fractal generator $g_i(\sigma)$ is p_i (for $i=1,2,\ldots,$ with $\sum_i p_i=1$). I generalize Eq. (1) to determine the dimension

$$\mathcal{G}(\sigma)=\sum\nolimits_i p_i g_i(\sigma)=1. \tag{2}$$

I call $\mathcal{G}(\sigma)$ the *genealogy* of the Cantor set.

We see that the probabilistic Cantor set describes the cross section of the attractor quite well. We can rewrite the genealogical equation (2) for the cross section of the attractor as

$$\mathcal{G}(\sigma)=\int_0^1 d\xi\, g_{01}(\sigma;\xi)=1, \tag{3}$$

where the subscripts correspond to the generations involved, and ξ corresponds to the line $x=\xi$. A random-phase assumption has allowed me to replace

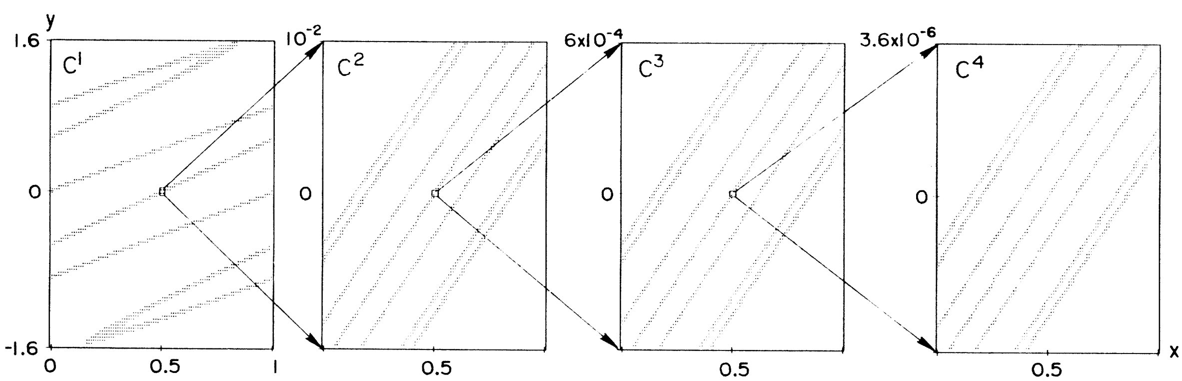

FIG. 1. Structures of the attractor analytically obtained for the dissipative standard map with $b=0.05$ and $k=1.5$: C^1 in the region $0\leq x\leq 1$, $|y|\leq 1.6$; C^2, C^3, and C^4 in regions $(0.5,0)\pm(10^{-2},10^{-2})$, $\pm(6\times10^{-4},6\times10^{-4})$, and $\pm(3.6\times10^{-6},3.6\times10^{-6})$. White region represents $f=0$ while black region represents $f>0$. Note the resemblance of the last three.

$\Sigma\Delta p$ by $\int d\xi$. The integrand can be found from V_0 (segment $[-\bar{y},\bar{y}]$) and the corresponding $V_1(\xi)$.

I have chosen ξ in the integral to be the abscissa variable. Different choices of coordinates ξ may lead to different genealogies. The best coordinate ξ should be chosen such that a curve of constant ξ represents points that are squeezed together upon application of the map. However, the solution σ of Eq. (3) has a rather insensitive dependence on the choice of coordinate ξ. Since the shrinking of a map is typically dominant along the ordinate axis,[21] choosing ξ to be the abscissa variable gives a reasonably accurate dimension as shown below.

For the Hénon map[22] ($x'=y+1-ax^2$, $y'=bx$) with $a=1.4$ and $b=0.3$, I choose the trapping region C^0 to be the quadrilateral with vertices at (−1.33, 0.42), (1.32, 0.133), (1.2525, −0.1127), and (−1.087, −0.408). Transforming C^0 into a rectangle and solving Eq. (3) [by first dividing the rectangle into a 100×100 grid and counting consecutive black cells (i.e., cells in C^1) on each column to obtain the genealogy numerically], I obtain $\sigma=0.258\pm0.005$.

For the Zaslavskii map[20]

$$x'=[x+\nu(1+\mu y)+\epsilon\nu\mu\cos 2\pi x](\mathrm{mod}1),$$

$$y'=\exp(-\Gamma)(y+\epsilon\cos 2\pi x),$$

where $\mu\equiv[1-\exp(-\Gamma)]/\Gamma$, with $\Gamma=3.0$, $\epsilon=0.3$, and $\nu=10^2\times\frac{4}{3}$, the trapping region C^0 is $|y|\le\bar{y}=\epsilon/(\exp\Gamma-1)=0.0157$. The first-order structures C^1 of the attractor are shown in Fig. 2. Consider vertical lines at intervals of 0.1. V_0 is $[-\bar{y},\bar{y}]$; V_1 consists of about fifty segments. If we divide the segment V_0 into 15 000 cells, we find that for the fractal generator of each x ($x=0,0.1,0.2,\ldots,0.9$), Eq. (1) has a solution of σ between 0.5595 and 0.5625. Noting that the variation in σ is so small with respect to x, it is safe to divide the trapping region into a 10×15 000 grid, and conclude that the solution to Eq. (3) is $\sigma=0.561\pm0.0015$.

For the Kaplan-Yorke map[9] [$x'=2x(\mathrm{mod}1)$, $y'=\alpha y+\cos 4\pi x$], with $\alpha=0.2$, I have $V_0=[-\bar{y},\bar{y}]$, $\bar{y}=1.25$. For any x, V_1 consists of two segments each of length 0.5. Although the two segments overlap in C^1, we are interested mainly in the fractal generators at generation infinity, where the region of overlapping (in C^∞) has measure zero. Hence, we can solve Eq. (1) exactly and find σ to be $\log(\frac{1}{2})/\log(0.5/2.5)=0.430\,676\,6$.

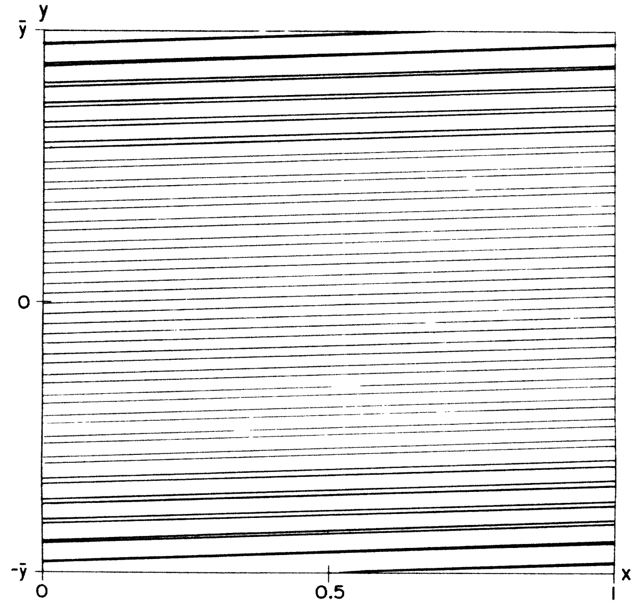

FIG. 2. First-order structures C^1 of the attractor for the Zaslavskii map with $\Gamma=3$, $\epsilon=0.3$, and $\nu=10^2\times\frac{4}{3}$.

In Table I, I summarize the results from the above method and from previous calculations.[10,11] The second column displays the values of d analytically determined as above, the third column shows values obtained from box-counting algorithms,[10] and the fourth column from the Kaplan-Yorke conjecture.[10] It is evident that the values of d determined analytically agree with those calculated with other methods. The calculation of dimensionality by use of the analytical method presented here is much faster than the other

TABLE I. Summary of results.

Map tested	d determined analytically	d from box counting[a]	d from Lyapunov numbers[b]
Hénon map, $a=1.4, b=0.3$	1.258 ± 0.005	1.261 ± 0.003	1.264 ± 0.002
Zaslavskii map, $\Gamma=3, \epsilon=0.3$, $\nu=10^2\times\frac{4}{3}$	1.561 ± 0.0015	···	1.55 ± 0.0005[b]
Kaplan-Yorke map, $\alpha=0.2$	1.430 676 6	1.4316 ± 0.0016	1.430 676 6

[a]Reference 10.
[b]Reference 11.

methods. As an example, to calculate the value of d for the Hénon attractor, according to Russell, Hanson, and Ott,[10] it took 5 min on the Cray computer using a box-counting algorithm, and 0.3 min using the Kaplan-Yorke conjecture. It takes only less than 0.02 min to determine d analytically by solving Eq. (3) numerically.

My results demonstrate that the genealogical equation (3) provides a simple, fast, and accurate analytical method for determining the dimensionality of strange attractors. The error bars in the second column of Table I are subjective estimations based on variation of grids. A more detailed analysis will be included in an expanded article in preparation.[23]

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[1]A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer-Verlag, New York, 1982), Chap. 7.

[2]B. B. Mandelbrot, *Fractals: Form, Chance and Dimension* (Freeman, San Francisco, 1977).

[3]P. Tabeling, Phys. Rev. A **31**, 3460 (1985).

[4]M. Bonetti, R. Meynart, and D. Olivari, Phys. Rev. Lett. **55**, 492 (1985).

[5]J. Guckenheimer and G. Buzyna, Phys. Rev. Lett. **51**, 1438 (1983).

[6]D. A. Weitz, J. S. Huang, M. Y. Lin, and J. Sung, Phys. Rev. Lett. **54**, 1416 (1985).

[7]A. Babloyantz, J. M. Salazar, and C. Nicolis, Phys. Lett. **111A**, 152 (1985).

[8]H. S. Greenside, A. Wolf, J. Swift, and T. Pignataro, Phys. Rev. A **25**, 3453 (1982).

[9]J. L. Kaplan and J. A. Yorke, in *Functional Differential Equations and Approximation of Fixed Points,* edited by H.-O. Peitgen and H.-O. Walther, Lecture Notes in Mathematics Vol. 730 (Springer-Verlag, New York, 1979), p. 204.

[10]D. A. Russell, J. D. Hanson, and E. Ott, Phys. Rev. Lett. **45**, 1175 (1980).

[11]R. Badii and A. Politi, Phys. Rev. Lett. **52**, 1661 (1984).

[12]Y. Termonia and Z. Alexandrowica, Phys. Rev. Lett. **51**, 1265 (1983).

[13]J. D. Farmer, E. Ott, and Y. A. Yorke, Physica **7D**, 153 (1983).

[14]J. D. Farmer, Z. Naturforsch. **37a**, 1304 (1982).

[15]In fact, $n(2^{-i}) = F_i$ (Fibonacci numbers with $F_0 = 1, F_1 = 2$). Capacity should be $\lim_{i\to\infty} \log F_i / \log 2^i = 0.6942$, same as the Hausdorff dimension.

[16]E. Ott, W. D. Withers, and J. A. Yorke, J. Stat. Phys. **36**, 687 (1984).

[17]K. Y. Tsang and M. A. Lieberman, Physica **11D**, 147 (1984).

[18]K. Y. Tsang and M. A. Lieberman, Phys. Lett. **103A**, 175 (1984).

[19]R. V. Jensen and C. R. Oberman, Physica **4D**, 183 (1982).

[20]G. M. Zaslavskii, Phys. Lett. **69A**, 145 (1978).

[21]A typical map is composed of three steps (not necessarily in the same order): kicking points at low y either by switching x and y or by deforming the plane; contracting along the y axis; stretching, shearing, or twisting along the x axis.

[22]M. Hénon, Commun. Math. Phys. **50**, 69 (1976).

[23]K. Y. Tsang, to be published.