Vacuum Space-Times That Admit No Maximal Slice

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Every closed three-manifold occurs as a spacelike hypersurface of a vacuum space-time. For most of these three-manifolds, however, the vacuum space-times that contain them have no maximal slice. For asymptotically flat manifolds there are no restrictions on which three-manifolds can occur obeying the local energy conditions $\rho > (J_a J^a)^{1/2}$, and the space-times that contain them in most cases have no maximal slice.

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The existence of maximal slices for physically reasonable space-times has been an outstanding problem in general relativity. Only partial results on existence have been proven to date.[1,2] In particular, space-times sufficiently near to Minkowski space admit maximal slices. On the other hand, Brill[3,4] gives examples of some spatially closed space-times and an asymptotically flat space-time that admits no maximal slices.

A less known problem is whether every three-manifold has physically reasonable initial data. Schoen and Yau[5] appear to claim that for asymptotically flat three-manifolds this is false, i.e., that if an asymptotically flat three-manifold has positive energy, then its compactification is the connected sum of three-manifolds with finite fundamental group and handles. As they note in a subsequent paper, however, the initial data in their classification were assumed to have no apparent horizons.[6] Hence, the problem of which three-manifolds can occur remains open. If the restriction on horizons is removed, Brill's example of an asymptotically flat space-time with no maximal slice shows that the class of allowed three-manifolds can be broadened, because the initial data for his example are on a three-torus minus a point. However, the dust source in the Brill example is not differentiable everywhere, and one might worry that in smoothing it one would be forced by the constraint equations to violate the local energy condition.

In what follows, it will be shown there are no restrictions on which closed or asymptotically flat three-manifolds have physically reasonable initial data.[7] In particular, every closed three-manifold has vacuum initial data. As a by-product of this work, it follows that most space-times with sources that obey the dominant energy condition never admit maximal slices. Moreover, this result is independent of the explicit form of the sources, and depends only on the topology of the space-time.

An initial data set in general relativity consists of a three-manifold Σ, Riemannian metric g_{ab}, symmetric tensor p_{ab} (which will be the extrinsic curvature of Σ in the space-time $R\times\Sigma$ evolved from the data), energy density ρ, and momentum density J^a which satisfy the constraints

$$R - p_{ab}p^{ab} + p^2 = 16\pi\rho, \qquad (1)$$

$$D_b(p^{ab} - pg^{ab}) = 8\pi J^a. \qquad (2)$$

Here R is the scalar curvature of g_{ab}, D_b is the covariant derivative defined by g_{ab}, and $p \equiv p_a{}^a$. Initial data are called physically reasonable if they are smooth (C^∞), Σ is geodesically complete with respect to g_{ab}, and the sources obey the local energy condition $\rho \geqslant (J_a J^a)^{1/2}$. From here on, initial data will always refer to physically reasonable data. When the energy and momentum densities correspond to classical nondissipative matter sources, or vacuum $\rho = J^a = 0$, the coupled Einstein-matter equations can be used to evolve the initial data into a space-time[1,8] $R\times\Sigma$. Moreover, Σ is a spacelike hypersurface in the evolved space-time, and the constraint equations and local energy condition are the orthogonal and parallel projections of the field equations and the dominant energy condition,[9] respectively.

It will now be shown that every nonnegative smooth function on a closed three-manifold is the energy density of an initial data set with $J^a = 0$, and in particular that every closed three-manifold has vacuum initial data. Let Σ be a closed three-manifold and $\rho \geqslant 0$ a smooth function on it. Since ρ is smooth and Σ closed, ρ attains a maximum on Σ denoted by ρ_0. Let $f \equiv 16\pi\rho - 6A_0^2$ where $6A_0^2 \equiv 16\pi\rho_0 + \epsilon$ and ϵ is any positive number. The function f is strictly negative on Σ. Now the following result of Kazdan and Warner[10] is applied: Given any closed manifold $M^n n \geqslant 3$ and smooth function which is negative somewhere on M^n, there exists a Riemannian metric with the prescribed function as its scalar curvature.

Choose g_{ab} on Σ such that its scalar curvature is f. Next take $p_{ab} = A_0 g_{ab}$ where $A_0 = [(16\pi\rho_0 + \epsilon)/6]^{1/2}$. It follows immediately that g_{ab}, p_{ab}, ρ, and $J^a = 0$ are initial data on Σ. Moreover, if ρ is taken to be the proper energy density of dust,[11] then the initial data can be evolved into a space-time $R\times\Sigma$ with Σ a spacelike hypersurface. In particular, when $\rho=0$, Σ is a spacelike hypersurface of a vacuum space-time.

Examples are the closed hyperbolic spaces. A hyperbolic space is obtained from R^3 with the metric $\mathbf{g}=dr^2/(1+k^2r^2)+r^2\,d\Omega^2$ (in spherical coordinates) by identifying points of R^3 via isometries of the metric. The scalar curvature of these spaces is $R=-6k^2$. A specific example is the hyperbolic dodecahedron space obtained from a solid dodecahedron by identifying opposite faces after a counterclockwise rotation of $3\pi/5$ radians. One choice of initial data on these spaces is g_{ab} with $R=-6k^2$, $p_{ab}=Ag_{ab}$, $J^a=0$, and $\rho=(3/8\pi)(A^2-k^2)$. If ρ is taken to be the proper energy density of dust, then the time evolution of the initial data is a Robertson-Walker space-time of negative spatial curvature.

The technique just applied to close three-manifolds is not a viable approach at the present time for Σ asymptotically flat for two reasons: The ability to prescribe scalar curvature with a complete metric is an open problem, and if p_{ab} is proportional to g_{ab} it will not approach zero at infinity. The procedure for asymptotically flat manifolds will be to satisfy the constraints on a closed manifold, then to remove a ball and smoothly glue to it a spacelike hypersurface of the Schwarzschild space-time.

The gluing procedure will be used to prove the following theorem, which is an existence theorem for asymptotically flat initial data on manifolds admitting a special type of metric.

Theorem.—Let S be a closed three-manifold with initial data. Suppose that in a neighborhood of some point i_0 of S the initial data satisfy the following conditions: The metric g_{ab} is spherically symmetric with scalar curvature $R=-6k^2$, $p_{ab}=Ag_{ab}$, $J^a=0$, and $\rho=(3/8\pi)(A^2-k^2)$. Then $S-\{i_0\}$ has asymptotically flat initial data.

Examples of closed three-manifolds with initial data satisfying the conditions of the above theorem are the closed hyperbolic and flat spaces. The standard metric on one of these spaces is spherically symmetric in some neighborhood of every point, and its scalar curvature is $-6k^2$ everywhere. Hence, initial data satisfying the conditions are the standard metric g_{ab}, $p_{ab}=Ag_{ab}$, $J^a=0$, and $\rho=(3/8\pi)(A^2-k^2)$. In particular, it follows that the three-torus T^3 minus a point has asymptotically flat initial data.

Now, the theorem will be proved. Local spherical symmetry about i_0 implies that the metric near i_0 can be written in the form $\mathbf{g}=\xi\,dr^2+r^2\,d\Omega^2$, where ξ is a function of only r and $r<r_2$ (r_2 fixed). Geodesic completeness and $R=-6k^2$ imply $\xi=(1+k^2r^2)^{-1}$. Next, this metric coefficient is matched to the $t=0$ spacelike hypersurface of the Schwarzschild space-time. Choose any r_0 and r_1 such that $r_2>r_1>r_0>(A^2-k^2)r_1^3$. Let η be any smooth monotonically decreasing function with $\eta=0$ for $r_2>r\geqslant r_1$ and $\eta=1$ for $r\leqslant r_0$. Now, make the following choices for $r_2>r\geqslant r_0$:

$$\xi=\left[1-\frac{2M(r)}{r}\right]^{-1}\eta+(1-\eta)(1+k^2r^2)^{-1},$$

$$\rho=(3/8\pi)(A^2-k^2)(1-\eta),\quad M_1=\frac{A^2-k^2}{2}r_1^3,$$

and

$$M(r)=M_1-\int_r^{r_1}4\pi\rho r^2\,dr.$$

For $r<r_0$, take g_{ab} to be the metric of the $t=0$ spacelike hypersurface of the maximally extended Schwarzschild space-time of mass $M(r_0)$. All the above functions are smooth, because they are constructed from integrals, products, and sums of smooth functions. Now, p_{ab} will be determined. Let p_{ab} be given by $\mathbf{p}=\alpha\,dr^2+\beta r^2\,d\Omega^2$ where α and β are functions of r. Integrating the constraint equations from r to r_1 with $J^a=0$ and the above choices of ξ and ρ yields

$$\beta^2=\frac{\xi^{-1}-[1-2M(r)/r]}{r^2}$$

and $\alpha=\xi(\beta'r+\beta)$. These expressions are smooth for $r_2>r\geqslant r_0$ by construction. When $r<r_0$ take p_{ab} for the Schwarzschild hypersurface. This completes the proof of the theorem.

It will now be shown that every closed three-manifold S has initial data satisfying the hypothesis of the above theorem. Let S be any closed three-manifold with metric g_{ab}. Take a small enough neighborhood N of a point i_0 of S such that N is diffeomorphic to the interior of a ball B of radius r_3 via a diffeomorphism $\psi:N\to B$. Let h_{ab} be the complete spherically symmetric metric of constant scalar curvature $-6k^2$ on B and $g_{ab}|_N$ the restriction of g_{ab} to $N\subset S$. Choose a smooth monotonically decreasing function η with $\eta=0$ for $r_3>r>r_2$ and $\eta=1$ for $r_2>r_1>r$. Let $\tilde{g}_{ab}$ be the metric on B defined by $\tilde{g}_{ab}\equiv(1-\eta)\psi^{-1}g_{ab}|_N+\eta h_{ab}$, where $\psi^{-1}g_{ab}|_N$ is the pullback of $g_{ab}|_N$ onto B. Finally, take $\bar{g}_{ab}$ on S to be g_{ab} on $S-N$ and $\psi\tilde{g}_{ab}$ on N. Then $\bar{g}_{ab}$ is a spherically symmetric metric of constant scalar curvature $-6k^2$ in a neighborhood of i_0.

Let $\bar{R}$ be the scalar curvature of $\bar{g}_{ab}$ on S. Because $\bar{R}$ is smooth and S closed, $\bar{R}$ attains a minimum on S denoted by $\bar{R}_0$. Let $\rho\equiv(\bar{R}+6A^2)/16\pi$ where $6A^2\equiv|\bar{R}_0|+\epsilon$, $\epsilon>0$. Since $\rho\geqslant0$, the choice $J^a=0$ means that the local energy condition is satisfied. Now, take $\bar{g}_{ab}$, $p_{ab}=A\bar{g}_{ab}$, $J^a=0$, and $\rho=(\bar{R}+6A^2)/16\pi$ as initial data on S. Applying the theorem in a neighborhood of i_0, it follows that $S-\{i_0\}$ has asymptotically flat initial data.

It was just shown that $S-\{\text{a point}\}$ has asymptotically flat initial data for every closed three-manifold S. Since every geodesically complete three-manifold with a single asymptotic region arises from the removal of a point from a closed three-manifold, it follows that there are no restrictions on which asymptotically flat three-manifold have initial data. Moreover, if ρ is taken to be the proper energy density of dust, then the initial data can be evolved into a space-time. Hence every asymptotically flat three-manifold is a spacelike hypersurface of a space-time with sources that obey the dominant energy condition.

Now, look at maximal slices for space-times $R\times\Sigma$ with Σ closed or asymptotically flat, and sources that obey the dominant energy condition. Suppose that the space-time has a maximal slice, that is, a spacelike hypersurface diffeomorphic to Σ with $p=0$. Then the constraints must be satisfied on Σ with $p=0$, and the dominant energy condition implies $\rho\geqslant(J_aJ^a)^{1/2}$. Constraint equation (1) combined with $p=0$ and $\rho\geqslant 0$ imply $R\geqslant 0$. Hence Σ admits a metric with $R\geqslant 0$. If Σ is asymptotically flat, then there is a closed three-manifold S such that S minus a point is Σ. Moreover, one can prove if Σ has a metric with $R\geqslant 0$, then S admits one with $R>0$.[12] Thus, the question of necessary conditions for existence of maximal slices is reduced to asking which closed three-manifolds have metrics with $R>0$ or $R\geqslant 0$. Gromov and Lawson[13] proved the following: Any closed oriented three-manifold X (or its double cover if nonorientable) which has a $K(\pi,1)$ [14] as prime factor in its prime decomposition[15] admits no metric with $R>0$. In fact, any metric on X with $R\geqslant 0$ is flat. Because most prime closed three-manifolds are $K(\pi,1)$'s and only ten of these are flat, it follows that most closed three-manifolds never admit metrics with $R\geqslant 0$. Therefore, most closed or asymptotically flat three-manifolds Σ never have a metric with $R\geqslant 0$. Since a necessary condition for maximal slice in $R\times\Sigma$ is that Σ admits a metric with $R\geqslant 0$, and all Σ occur, it follows that most space-times $R\times\Sigma$ with Σ closed or asymptotically flat and sources obeying the dominant energy never admit a maximal slice.[16]

The surprising feature of the counterexamples presented here is that they are independent of the explicit form of the sources. Furthermore, the weak condition for existence of maximal slices in asymptotically flat space-times given by Bartnick[2] never can be satisfied on most space-times.

The results presented here for three-manifolds with a single asymptotic region are easily extended to manifolds with any finite number of asymptotic regions by repeating the gluing procedure at a finite number of points of a closed three-manifold. For closed n-manifolds in $(n+1)$-dimensional gravity the constraint equations are the same as (1) and (2). They can be satisfied by a change of the proportionality constant A_0 to $[(16\pi\rho_0+\epsilon)/(n^2-n)]^{1/2}$. The maximal-slice results also carry over to higher dimensions.

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[1] Y. Choquet-Bruhat and J. W. York, Jr., in *General Relativity and Gravitation,* edited by A. Held (Plenum, New York, 1980), Vol. 1. See also references cited therein.

[2] R. Bartnick, in *Asymptotic Behavior of Mass and Spacetime Geometry,* edited by F. J. Flaherty (Springer-Verlag, New York, 1984).

[3] D. R. Brill, in *Proceedings of the First Marcel Grossmann Meeting on General Relativity,* edited by R. Ruffini (North-Holland, New York, 1977).

[4] D. R. Brill, in *Proceedings of the Third Marcel Grossmann Meeting on the Recent Developments of General Relativity,* edited by Hu Ning (North-Holland, New York, 1983).

[5] R. Schoen and S. T. Yau, Phys. Rev. Lett. **43**, 1457 (1979); S. T. Yau, in *The Chern Symposium 1979,* edited by W. T. Hsiang *et al.* (Springer-Verlag, New York, 1980).

[6] R. Schoen and S. T. Yau, Commun. Math. Phys. **90**, 575 (1983).

[7] A compact manifold without boundary is called *closed.* A three-manifold is *asymptotically flat* with a single asymptotic region if for some compact set K, $\Sigma-K$ is connected and diffeomorphic to R^3 minus a ball B. The metric g_{ab} and extrinsic curvature p_{ab} on $\Sigma-K$ are required to satisfy $\hat{g}_{ab}-\delta_{ab}=O(1/r)$, $\partial_c\hat{g}_{ab}=O(1/r^2)$, $\partial_d\partial_c\hat{g}_{ab}=O(1/r^3)$, $\hat{p}_{ab}=O(1/r^2)$, and $\partial_c\hat{p}_{ab}=O(1/r^3)$ where $\hat{g}_{ab}$ and $\hat{p}_{ab}$ are the pullbacks onto R^3-B.

[8] S. W. Hawking and G. F. Ellis, *The Large Scale Structure of Spacetime* (Cambridge Univ. Press, Cambridge, England, 1973).

[9] Sources obey dominant energy condition if their stress-energy tensor satisfies $T_{\alpha\beta}W^\alpha W^\beta\geqslant 0$ and $T_{\alpha\beta}W^\beta T^\alpha_\gamma W^\gamma\leqslant 0$ for all W^α on the space-time with $W_\alpha W^\alpha<0$.

[10] J. Kazdan and F. Warner, J. Differential Geom. **10**, 113 (1975).

[11] The stress-energy tensor for dust is $T_{\alpha\beta}=\rho u_\alpha u_\beta$ where $u_\alpha u^\alpha=-1$.

[12] The compactification admitting a metric with positive scalar curvature is hinted at in Ref. 4, but no proof is offered. The details of the proof will appear in D. M. Witt, to be published.

[13] M. Gromov and H. B. Lawson, Jr., Inst. Hautes Etudes Sci. Publ. Math. **58**, 83 (1983). See also the earlier work of Schoen and Yau for partial results: R. Schoen and S. T. Yau, Ann. of Math. **110**, 127 (1979), and Manuscripta Math. **28**, 159 (1979).

[14] A $K(\pi,1)$ is a closed three-manifold with a universal covering manifold which is deformable to point. See J. Hempel, *3-Manifolds* (Princeton Univ. Press, Princeton, NJ, 1976).

[15]The connected sum of two three-manifolds M_1 and M_2 is the three-manifold $M_1 \# M_2$ obtained from removing the interior of a ball from each, and then gluing the resulting manifolds together along their boundaries. A three-manifold M is *prime* if $M = M_1 \# M_2$ implies that M_1 or M_2 is a three-sphere. Every closed orientable three-manifold M has a unique *prime decomposition* $M = \#_{i=1}^{k} M_i$ where each M_i is prime.

[16]The proof of this result used only the *weak energy condition,* $T_{\alpha\beta} W^\alpha W^\beta \geq 0$ for all $W_\alpha W^\alpha < 0$. Therefore, this result extends to the larger class of space-times obeying the weak energy condition.