Nonadiabatic Level Crossing in Resonant Neutrino Oscillations

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Analytic results are presented for the probability of detecting an electron neutrino after passage through a resonant oscillation region. If the electron neutrino is produced far above the resonance density, this probability is simply given by $\langle P_{\nu_e} \rangle \approx \sin^2\theta_0 + P_x \cos2\theta_0$, where θ_0 is the vacuum mixing angle. The probability is averaged over the production as well as the detection positions of the neutrino and P_x is the Landau-Zener transition probability between adiabatic states. Finally, this result is applied to resonance oscillations within the solar interior.

Recently Mikheyev and Smirnov[1] and Bethe[2] have revived interest in the solar-neutrino deficit by demonstrating that electron neutrinos produced in the sun can be efficiently rotated into muon neutrinos by passage through a resonant oscillation region. This mechanism may solve the solar-neutrino puzzle. In this paper, I present an analytic result for the probability of detecting an electron neutrino after passage through one or more resonant oscillation regions. This result is then used to show the regions of parameter space, the difference of the squared masses versus the vacuum mixing angle, for which the solar-neutrino puzzle is solved.

A neutrino state is assumed to be a linear combination of the two flavor states $|\nu_e\rangle$ and $|\nu_\mu\rangle$:

$$|\nu,t\rangle = C_e(t)|\nu_e\rangle + C_\mu(t)|\nu_\mu\rangle. \tag{1}$$

If the neutrinos are massive, then the mass eigenstates need not be identical to the flavor eigenstates, so that the Dirac equation which governs the evolution of the neutrino state is not necessarily diagonal in the flavor basis. This leads to the well known phenomena of vacuum neutrino oscillations. In the presence of matter, the nondiagonal nature of this evolution is further enhanced by coherent forward scattering which can lead to resonant neutrino oscillations. Wolfenstein[3] has derived the Dirac equation for this process, in the ultrarelativistic limit, in terms of the vacuum mass eigenstates. Here, I use his result, in the flavor basis, after discarding a term proportional to the identity matrix, as this term only contributes an overall phase factor to the state $|\nu,t\rangle$. The resulting Schrödinger-type wave equation is

$$i\frac{d}{dt}\begin{bmatrix} C_e \\ C_\mu \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -\Delta_0\cos2\theta_0 + \sqrt{2}G_F N & \Delta_0\sin2\theta_0 \\ \Delta_0\sin2\theta_0 & \Delta_0\cos2\theta_0 - \sqrt{2}G_F N \end{bmatrix}\begin{bmatrix} C_e \\ C_\mu \end{bmatrix}, \tag{2}$$

where $\Delta_0 = (m_2^2 - m_1^2)/2k$, $m_{1,2}$ are the neutrino masses, k is the neutrino momentum, θ_0 is the vacuum mixing angle, N is number density of electrons, and G_F is the Fermi constant. The constraints $\Delta_0 > 0$ and $\theta_0 < \pi/4$ are assumed. At an electron density, N, the matter mass eigenstates are

$$|\nu_1, N\rangle = \cos\theta_N|\nu_e\rangle - \sin\theta_N|\nu_\mu\rangle,$$
$$|\nu_2, N\rangle = \sin\theta_N|\nu_e\rangle + \cos\theta_N|\nu_\mu\rangle, \tag{3}$$

which have eigenvalues $\pm\Delta_N/2$, where

$$\Delta_N = [(\Delta_0\cos2\theta_0 - \sqrt{2}G_F N)^2 + \Delta_0^2\sin^2 2\theta_0]^{1/2}, \tag{4}$$

and θ_N satisfies

$$\Delta_N\sin2\theta_N = \Delta_0\sin2\theta_0. \tag{5}$$

These states evolve in time by the multiplication of a phase factor, if the electron density is a constant. For such a constant density there are three regions of interest: (i) Well below resonance, $\sqrt{2}G_F N \ll \Delta_0\cos2\theta_0$, where the matter mixing angle is $\theta_N \sim \theta_0$ and the oscillation length is $L_0 = 2\pi/\Delta_0$. Typically, this is the region that the electron neutrinos are detected in. (ii) At resonance, $\sqrt{2}G_F N = \Delta_0\cos2\theta_0$, where the matter mixing angle is $\theta_N = \pi/4$ and the resonant oscillation length is $L_R = L_0/\sin2\theta_0$, which for small vacuum mixing angle can be many times the vacuum oscillation length. (iii) Far above resonance, $\sqrt{2}G_F N \gg \Delta_0\cos2\theta_0$, where the matter mixing angle $\theta_N \sim \pi/2$, and the oscillation length $L_N = 2\pi/\Delta_N$ is much smaller than the vacuum oscillation length L_0. For the situation of current interest the electron neu-

trinos are produced above resonance, pass through resonance, and are detected in the vacuum.

If the electron density varies slowly, the states which evolve independently in time (the adiabatic states) are

$$\exp(-i\tfrac{1}{2}\int^t \Delta_N \, dt)|\nu_1, N(t)\rangle$$

and

$$\exp(+i\tfrac{1}{2}\int^t \Delta_N \, dt)|\nu_2, N(t)\rangle.$$

Therefore, it is convenient to use these states, as the basis states, in the region for which there are no transitions (away from the resonance region). As a neutrino goes through resonance these adiabatic states may be mixed, but on the other side of resonance, the neutrino state can still be written as a linear combination of these states. That is, a basis state produced at time t, going through resonance at time t_r, and detected at time t' is described by

$$\exp(-i\tfrac{1}{2}\int_{t_r}^{t} \Delta_N \, dt)|\nu_1, N(t)\rangle \to a_1 \exp(-i\tfrac{1}{2}\int_{t_r}^{t'} \Delta_N \, dt)|\nu_1, N(t')\rangle + a_2 \exp(+i\tfrac{1}{2}\int_{t_r}^{t'} \Delta_N \, dt)|\nu_2, N(t')\rangle,$$

$$\exp(+i\tfrac{1}{2}\int_{t_r}^{t} \Delta_N \, dt)|\nu_2, N(t)\rangle \to -a_2^* \exp(-i\tfrac{1}{2}\int_{t_r}^{t'} \Delta_N \, dt)|\nu_1, N(t')\rangle + a_1^* \exp(+i\tfrac{1}{2}\int_{t_r}^{t'} \Delta_N \, dt)|\nu_2, N(t')\rangle,$$

where a_1 and a_2 are complex numbers such that $|a_1|^2 + |a_2|^2 = 1$. The relationship between the coefficients, for these two basis states, is due to the special nature of the wave equation, Eq. (2). The phase factors have been chosen so that coefficients a_1 and a_2 are characteristics of the transitions at resonance and are not related to the production and detection of the neutrino state.

Hence, the amplitude for producing, at time t, and detecting, at time t', an electron neutrino after passage through resonance is

$$A_1(t)\exp(-i\tfrac{1}{2}\int_{t_r}^{t'} \Delta_N \, dt) + A_2(t)\exp(+i\tfrac{1}{2}\int_{t_r}^{t'} \Delta_N \, dt),$$

where

$$A_1(t) = \cos\theta_0[a_1 \cos\theta_N \exp(+i\tfrac{1}{2}\int_{t_r}^{t} \Delta_N \, dt) - a_2^* \sin\theta_N \exp(-i\tfrac{1}{2}\int_{t_r}^{t} \Delta_N \, dt)],$$

$$A_2(t) = \sin\theta_0[a_2 \cos\theta_N \exp(+i\tfrac{1}{2}\int_{t_r}^{t} \Delta_N \, dt) + a_1^* \sin\theta_N \exp(-i\tfrac{1}{2}\int_{t_r}^{t} \Delta_N \, dt)].$$

Thus the probability of detecting this neutrino as an electron neutrino is given by

$$P_{\nu_e}(t,t') = |A_1(t)|^2 + |A_2(t)|^2 + 2|A_1(t)A_2(t)|\cos(\int_{t_r}^{t'} \Delta_N \, dt + \Omega)$$

with $\Omega = \arg(A_1^* A_2)$. After averaging over the detection position, the detection averaged probability is given by

$$P_{\nu_e}(t) = \tfrac{1}{2} + \tfrac{1}{2}(|a_1|^2 - |a_2|^2)\cos2\theta_N \cos2\theta_0 - |a_1 a_2|\sin2\theta_N \cos2\theta_0 \cos(\int_{t_r}^{t} \Delta_N \, dt + \omega)$$

with $\omega = \arg(a_1 a_2)$. The last term shows that the phase of the neutrino oscillation at the point the neutrino enters resonance can substantially affect this probability. Therefore, we must also average over the production position to obtain the fully averaged probability of detecting an electron neutrino as

$$\langle P_{\nu_e} \rangle = \tfrac{1}{2} + (\tfrac{1}{2} - P_x)\cos2\theta_N \cos2\theta_0, \tag{6}$$

where $P_x = |a_2|^2$, the probability of transition from $|\nu_1, N\rangle$ to $|\nu_2, N\rangle$ (or vice versa) during resonance crossing. The adiabatic case[4] is trivially obtained by setting $P_x = 0$. Also, if the electron neutrinos are produced at a density much greater than the resonance density, so that $\cos2\theta_N \sim -1$, then

$$\langle P_{\nu_e} \rangle \approx \sin^2\theta_0 + P_x \cos2\theta_0. \tag{7}$$

Thus for small θ_0 the probability is just equal to the probability of level crossing during resonance passage.

Similar calculations can also be performed for the case of double resonance crossing (neutrinos from the far side of the sun). Here we must average not only over the production and detection positions of the neutrino but also over the separation between resonances. This sensitivity to the separation of the resonances can be understood as the effect of the phase of the oscillation as the neutrino enters the second resonance region. The fully averaged probability of detecting an electron neutrino is the same as Eq. (6) with P_x replaced by $P_{1x}(1 - P_{2x}) + (1 - P_{1x})P_{2x}$ (the classical probability result). Therefore, the generalization to any number of resonance regions, suitably averaged, is obvious.

To calculate the probability P_x, I make the approximation that the density of electrons varies linearly in the transition region. That is, a Taylor-series expan-

sion is made about the resonance position and the second- and higher-derivative terms are discarded;

$$N(t) \approx N(t_r) + (t - t_r) \, dN/dt|_{t_r}. \tag{8}$$

In this approximation the probability of transition between adiabatic states was calculated by Landau and Zenner.[5] This is achieved by solving the Schrödinger equation, Eq. (2), exactly in this limit. The solution is in terms of Weber (parabolic cylinder) functions. Application of the Landau-Zenner result to the current situation gives

$$P_x = \exp\left[-\frac{\pi}{2} \frac{\sin^2 2\theta_0}{\cos 2\theta_0} \frac{\Delta_0}{|(1/N) \, dN/dt|_{t_r}} \right]. \tag{9}$$

This expression, together with Eq. (6), are the main analytical results of this paper and demonstrate that only the electron number density, at production, and the logarithmic derivative of this density, at resonance, determine the probability of detecting an electron neutrino in the vacuum. It should be emphasized here that this result assumes that the neutrino state is produced before significant transitions take place and thus Eq. (9) is not valid for neutrinos produced in the transition region.

From Eq. (9) the size of the transition region can be determined. There are significant transitions ($P_x > 0.01$) if $\theta_0 < \theta_{\text{crit}}$, where θ_{crit} satisfies

$$\frac{\sin^2 2\theta_{\text{crit}}}{\cos 2\theta_{\text{crit}}} = 3 \frac{1}{\Delta_0} \left| \frac{1}{N} \frac{dN}{dt} \right|_{t_r}. \tag{10}$$

Hence, the maximum separation between the eigenstates for which transitions take place is $\Delta_0 \sin 2\theta_{\text{crit}}$. Therefore, the transition region is defined by

$$\Delta_N < \Delta_0 \sin 2\theta_{\text{crit}}. \tag{11}$$

This can only happen if $\theta_0 < \theta_{\text{crit}}$. In this transition region, the maximum variation of the electron number density from the resonant value is $\pm \delta N$, where

$$\delta N / N(t_r) = \sin 2\theta_{\text{crit}}.$$

Thus, the size of the transition region is

$$|t - t_r| = \frac{\sin 2\theta_{\text{crit}}}{|(1/N) \, dN/dt|_{\text{tr}}}.$$

This is the maximum $|t - t_r|$ for which the linear approximation must be good, so that Eq. (9) gives a reasonable estimate of the probability of crossing. For an exponential density profile, the Taylor-series expansion is an expansion in $\sin 2\theta_{\text{crit}}$, so that for small θ_{crit} this is an excellent approximation.

For the sun, the density profile is exponential except for the region near the center. In Fig. 1, I have plotted the probability contours for detection of an electron neutrino at the Earth in the $\Delta_0/\sqrt{2} G_F N_c$ vs $\sin 2\theta_0$
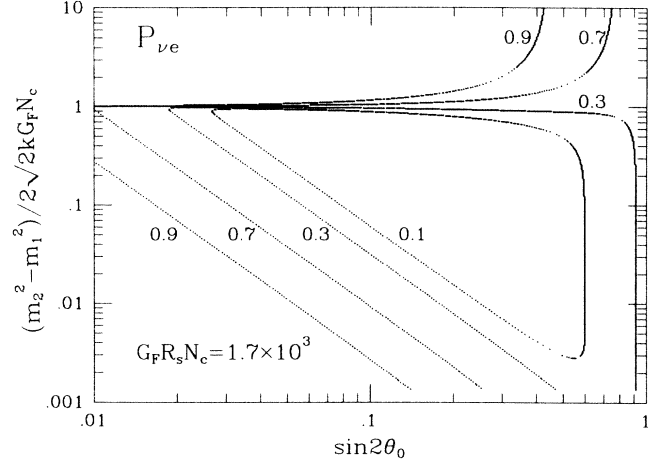

FIG. 1. Probability-contour plot for detecting an electron neutrino at the Earth which was produced in the solar interior.

plane for an exponential density profile. N_c is the electron number density at the point at which the electron neutrinos are produced. This plot depends only on the properties of the sun and this dependency is only through the combination $R_s N_c$, where R_s is the scale height. For Fig. 1, I have used an N_c corresponding to a density of 140 g/cm^3 and $Y_e = 0.7$. The scale height R_s is 0.092 times the radius of the sun.

Above the line $\Delta_0/\sqrt{2} G_F N_c = 1/\cos 2\theta_0$, the neutrinos never cross the resonance density on their way out of the sun. Here, the probability of detecting an electron neutrino is close to the standard neutrino-oscillation result. Below this line, the effects of passing through resonance come into play. Inside the 0.1 contour, there is only a small probability of transitions between the adiabatic states as the neutrino passes through resonance. To the right-hand side of this contour, the probability of detecting a neutrino grows, not because of transitions, but because both adiabatic states have a substantial mixture of electron neutrino at zero density. To the left-hand side and below this contour, the probability grows because here there are significant transitions between the adiabatic states as the neutrino crosses resonance. The diagonal lines of these contours have slope of -2 because of the form of P_x. It is only the intercept of these lines which depends on the product $R_s N_c$. Therefore, if one wishes to change the production density, which is held fixed in this plot, only these lines need to be shifted. In fact, a line labeled with P_e "crosses" $\Delta_0/\sqrt{2} G_F N_c = 1$ when a small θ_0 satisfies

$$\frac{\sin^2 2\theta_0}{\cos 2\theta_0} = \frac{-\sqrt{2} \ln P_e}{\pi G_F R_s N_c}. \tag{12}$$

Note that I find the probability of detecting an electron

neutrino, which crosses resonance, to be greater than 0.25 when $\theta_0 < 0.01$.

This isoprobability plot can easily be converted into an approximate iso-SNU (solar neutrino units) plot for the Davis et al. experiment.[6] The predicted result for this experiment[7] is 6 SNU, with 4.3 SNU coming from the ^{8}B neutrinos and 1.6 SNU from the lower-energy neutrinos (pep, ^{7}Be, ^{13}N, and ^{15}O), whereas Davis et al. observe 2.1 ± 0.3 SNU. Roughly speaking, the 2 SNU contour, in the $m_2^2 - m_1^2$ vs $\sin 2\theta_0$ log-log plot, will be a triangle, similar to the 0.3 contour of Fig. 1, with rounded corners. The three straight sections of this triangle are approximately given below. The horizontal line is given by choosing the parameters so that all the low-energy neutrinos and only 12% of the ^{8}B neutrinos are observed. This gives the constraints obtained by Bethe,[2]

$$m_2^2 - m_1^2 \approx 8 \times 10^{-5} \text{ eV}^2,$$
$$0.03 < \sin 2\theta_0 < 0.6. \tag{13}$$

For the vertical line, the probability of detecting an electron neutrino is nearly independent of energy, if $1 > \Delta_0/\sqrt{2} G_F N_c > 10^{-3}$. Therefore, we need to reduce all neutrinos by 30%.[4] This is achieved when

$$8 \times 10^{-8} \text{ eV}^2 < m_2^2 - m_1^2 < 1 \times 10^{-5} \text{ eV}^2,$$
$$\sin 2\theta_0 \approx 0.9. \tag{14}$$

For the diagonal line, we need to arrange that the Davis experiment only observed 50% of the ^{8}B neutrinos and none of the lower-energy neutrinos.[8,9] This is achieved when the probability for the mean ^{8}B neutrino, weighted by the detector cross section (energy ~ 10 MeV), is 0.5. This gives the following constraint:

$$(m_2^2 - m_1^2)\sin^2 2\theta_0 \approx 3 \times 10^{-8} \text{ eV}^2,$$
$$0.03 < \sin 2\theta_0 < 0.6. \tag{15}$$

To summarize, Eqs. (13)–(15) give regions of parameter space for which the expected result from the Davis experiment is ~ 2 SNU.

Since the proposed gallium experiment observes lower-energy neutrinos, from the pp process, these three regions will be distinguishable by use of the results of this experiment. More precise iso-SNU plots, for both experiments, are being generated taking into account the production energy and production position distributions of the neutrinos from the various processes within the solar interior.

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