Derivative Expansion for the One-Loop Effective Actions with Internal Symmetry

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A simple systematic method is presented for the evaluation of the derivative expansion of the functional determinant with covariant differential operators, space-time–dependent background fields, and internal symmetry. The results are directly applicable to the one-loop perturbative effective-action expansion for bosons and fermions. Derivative expansions up to four-derivative terms for $\mathrm{Tr}\ln[-\Pi^2+U(X)]$ and the effective action of the SO(N) linear σ model are calculated.

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Recently, there has been increasing interest in effective field theories with higher-derivative couplings. Examples are the ordinary and supersymmetric nonlinear σ model, the strongly interacting heavy Higgs sector of the standard model, skyrmion physics, and vacuum tunneling. This enhances the urgency to solve the important problem of developing a simple systematic procedure for the evaluation of the derivative expansion for the effective action and the operator expansion. A recent series of papers has taken the first step toward this goal.[1–6] The purpose of this paper is to present the best possible method applicable to any model which may contain internal symmetry, derivative couplings, gauge fields, and other background fields.

The objective is to incorporate the short-distance effects of the unobservable fields into an effective action of observable fields. As suggested in Ref. 1, an efficient way to accomplish this objective is to integrate out the unobservable quantum degree of freedom perturbatively. One can make use of the finite translation property to carry out the derivative expansion in such a way that the quantum corrections can be calculated in momentum space with use of modified Feynman propagators while the space-time–dependent observable background fields are kept intact in the configuration space for the final expression of the effective action. However, because of the short-distance singularity of the field theory, the expansion cannot be performed in a straightforward manner, but rather through an implicit regularization scheme. It is necessary to functionally differentiate the trace logarithm function before the expansion and subsequently undo it by a functional integration.[1] Other regularization schemes do not simplify the procedures substantially.[2,4] Moreover, the direct generalization of this method to include covariant derivative couplings and internal symmetry is rather complicated.[4] I report in this paper a novel method which would retain all the advantages of the previous method and at the same time eliminate these problems. Since the main idea can be naturally generalized to high-order perturbation calculations, this presentation will be limited to the one-loop correction. After a general description of the method, I shall apply it to calculate the derivative expansions of $\mathrm{Tr}\ln[-\Pi^2+U(X)]$ and the effective action of the linear SO(N) σ model.

The one-loop contribution to the effective action has the general form[7]

$$\mathrm{Tr}\ln G^{-1}(\Pi_\mu,U(X)), \tag{1}$$

where $U(X)$ represents a set of background fields and is a matrix in the coordinate space, internal symmetry space, and spin space. $(\Pi_\mu)_{ij}=\delta_{ij}P_\mu-V_\mu^a(X)t_{ij}^a$ is the generalized momentum and V_μ^a are the gauge fields while t_{ij}^a are the generators of the gauge group. The coordinate matrix elements are $\langle x|U_{ij}(X)|y\rangle=U_{ij}(x)\delta^D(x-y)$ and $\langle x|P^\mu|y\rangle=-i\partial_x^\mu\delta^D(x-y)$. The calculation will be performed in the D-dimensional Euclidean space.

The essential ingredient of this calculation is the nonvanishing commutator $[\Pi_\mu,X_\nu]=[P_\mu,X_\nu]=-i\delta_{\mu\nu}$. The covariant derivatives defined by $\mathcal{D}_\mu U(X)=i[\Pi_\mu,U(X)]$, $F_{\mu\nu}(X)=-i[\Pi_\mu,\Pi_\nu]$, and their subsequent covariant derivatives $\mathcal{D}_\mu\mathcal{O}(X)=i[\Pi_\mu,\mathcal{O}(X)]$ are functions of X only. Therefore, if one can use the cyclic permutation property of the trace operation and express $\mathrm{Tr}\ln G^{-1}$ as the trace of a function of covariant derivatives then

$$\mathrm{Tr}\ln G^{-1}=\mathrm{Tr}f(X)=\int d^Dx\,\langle y|f(X)|x\rangle|_{y=x}$$
$$=\int d^Dx\,f(x)\delta^D(0).$$

The presence of the infinite factor $\delta^D(0)$ implies that some kind of regularization must be used to provide a natural cancellation of the factor $\delta^D(0)$ to yield a finite result. As described briefly above there are usually unwanted complications associated with any regularization scheme. However, in this case there is a natural and physical averaging process already present in any loop calculation which would provide such service with no additional cost, namely the average over the internal loop momentum of the Feynman graph. The crucial development of this work is the recognition that this average procedure can be implemented at the very beginning stage in the presence of the background fields. The advantages for doing so will be apparent later.

Equation (1) is invariant under a finite momentum translation,

$$\mathrm{Tr}\ln G^{-1}(\Pi_\mu, U(X)) = \mathrm{Tr}\, e^{ip\cdot X}\ln G^{-1}(\Pi_\mu, U(X))e^{-ip\cdot X} = \mathrm{Tr}\ln G^{-1}(\Pi_\mu + p_\mu, U(X)). \quad (2)$$

The arbitrary momentum p_μ can be averaged over the entire momentum space,

$$\mathrm{Tr}\ln G^{-1}(\Pi_\mu, U(X)) = \frac{1}{\delta^D(0)}\int\frac{d^Dp}{(2\pi)^D}\mathrm{Tr}\ln G^{-1}(\Pi_\mu + p_u, U(X)), \quad (3)$$

where

$$\delta^D(0) = \int d^Dp/(2\pi)^D = V_p/(2\pi)^D$$

is exactly the infinite factor required for the regularization to work. The expression in Eq. (3) is the generating functional for the n-point vertex functions. p_μ is the loop momentum while Π_μ carries the momentum of the external fields.

In addition to fulfilling the regularization function, the introduction of the momentum integration without disturbing the full trace operation offers the needed freedom for manipulations, such as cyclic permutations of the operators and integrations by parts, in order to bring $\mathrm{Tr}\ln G^{-1}$ into the trace of a function of the covariant derivatives in the covariant derivative expansion. The procedure for the covariant derivative expansion becomes exceedingly simple. First, one expands Eq. (3) in a power series of Π_μ,

$$\mathrm{Tr}\ln G^{-1}(\Pi_\mu, U) = \frac{1}{\delta^D(0)}\int\frac{d^Dp}{(2\pi)^D}\Big\{\mathrm{Tr}\ln G^{-1}(p_\mu, U(X)) - \mathrm{Tr}\sum_{n=1}^{\infty}\frac{1}{n}[-G(p_\mu, U)(G^{-1}(\Pi_\mu + p_\mu, U) - G^{-1}(p_\mu, U))]^n\Big\}, \quad (4)$$

which can be manipulated into functions of covariant derivatives with no other isolated Π_μ operator. The trace can then be evaluated easily in the coordinate space to give the final result.

Practically all boson-loop and fermion-loop contributions excluding the anomaly contribution can be cast into the general form

$$G^{-1}(\Pi_\mu, U(X)) = \Pi^2 + U(X) \quad (5)$$

and

$$G^{-1}(\Pi_\mu + p_\mu, U(X)) = \Delta^{-1} + \Pi^2 + 2p\cdot\Pi, \quad (6)$$

with

$$\Delta^{-1}(X) = p^2 + U(X) = G^{-1}(p_\mu, U(X)). \quad (7)$$

For a second-order operator, it is more convenient to carry out the power expansion in two steps:

$$\mathrm{Tr}\ln(\Pi^2 + U) = \frac{1}{\delta^D(0)}\int\frac{d^Dp}{(2\pi)^D}\ldots\mathrm{Tr}\{\ln(p_\mu^2 + U) + \ln(1+\Delta\Pi^2) + \ln[1 + (1+\Delta\Pi^2)^{-1}\Delta 2p\cdot\Pi]\}. \quad (8)$$

For the second term, a unity factor $1 = (1/D)(\partial/\partial p_\mu)p_\mu$ can be inserted into the integrand and an integration by parts yields $(2/D)p^2(1+\Delta\Pi^2)^{-1}\Delta^2\Pi^2$. The expansion of the third term gives

$$-\sum_{n=1}^{\infty}\frac{1}{n}[-(1+\Delta\Pi^2)^{-1}\Delta 2p\cdot\Pi]^n.$$

Performing the angular average and including terms through Π^4, one obtains

$$\mathrm{Tr}\ln(\Pi^2 + U) = \frac{1}{\delta^D(0)}\int\frac{d^Dp}{(2\pi)^D}\mathrm{Tr}\Big\{\ln(p^2+U) + \frac{2}{D}p^2[\Delta^2\Pi^2 - \Delta\Pi^\mu\Delta\Pi_\mu] - \frac{2}{D}p^2(\Delta\Pi^2\Delta^2\Pi^2 - 2\Delta\Pi^2\Delta\Pi^\mu\Delta\Pi_\mu) - \frac{4}{D(D+2)}p^4[2\Delta\Pi_\mu\Delta\Pi^\mu\Delta\Pi_\nu\Delta\Pi^\nu + \Delta\Pi_\mu\Delta\Pi_\nu\Delta\Pi^\mu\Delta\Pi^\nu]\Big\}. \quad (9)$$

It is not difficult to recognize that $\mathrm{Tr}[\Delta^2\Pi^2 - \Delta\Pi^\mu\Delta\Pi_\mu] = -\frac{1}{2}\mathrm{Tr}[\Pi^\mu,\Delta]^2 = \frac{1}{2}\mathrm{Tr}(\mathcal{D}^\mu\Delta)^2$. With the identity $p^2 = [1/(D+2)](\partial/\partial p_\mu)(p_\mu p^2)$ and an integration by parts, the two $O(\Pi^4)$ terms can be combined ,

$$-\frac{4}{D(D+2)}p^4\,\mathrm{Tr}[2\Delta^3\Pi^2\Delta\Pi^2 + \Delta^2\Pi^2\Delta^2\Pi^2 - 2\Delta^2\Pi^\mu\Delta\Pi_\mu\Delta\Pi^2 - 2\Delta\Pi^\mu\Delta^2\Pi_\mu\Delta\Pi^2 - 2\Delta\Pi^\mu\Delta\Pi_\mu\Delta^2\Pi^2 + 2(\Delta\Pi^\mu\Delta\Pi_\mu)^2 + (\Delta\Pi^\mu\Delta\Pi^\nu)^2]. \quad (10)$$

Making full use of the cyclic permutation freedom and with some algebraic manipulation, one can express this $O(\Pi^4)$ contribution in terms of the commutators of Π_μ or the covariant derivatives. It becomes straightforward to take the trace in the coordinate space and complete the calculation. Returning to the Minkowski space but keeping the momentum explicitly in Euclidean space we have the final result

$$\mathrm{Tr}\ln(-\Pi^2+U(X))$$
$$=i\int d^Dx\int\frac{d^Dp_E}{(2\pi)^D}\mathrm{tr}\Big\{\ln[p_E^2+U(x)]-\frac{1}{D}p_E^2(\mathcal{D}_\mu\Delta)^2-\frac{2}{D(D+2)}p_E^4\{2[\Delta(\mathcal{D}^2\Delta)]^2+[(\mathcal{D}^\mu\Delta)(\mathcal{D}^\nu\Delta)]^2$$
$$-2[(\mathcal{D}^\mu\Delta)(\mathcal{D}_\mu\Delta)]^2-F^{\mu\nu}\Delta^2F_{\mu\nu}\Delta^2-4iF^{\mu\nu}\Delta(\mathcal{D}_\mu\Delta)\Delta(\mathcal{D}_\nu\Delta)\}\Big\},\quad(11)$$

where $\Delta(x)=1/[p_E^2+U(x)]$. The trace tr is for the internal symmetry and spin spaces only.

In $D=4$ dimensions, the first term gives the well-known contribution to the effective potential and is quadratically divergent.[1,8] We shall use dimensional regularization since the quadratically divergent contribution can always be recovered by applying to the result other regularization procedures such as described in Ref. 4. The $O(F_{\mu\nu}^2)$ term is logarithmically divergent. All other terms in the derivative expansion are finite. In the absence of any internal symmetry and gauge field, Eq. (11) reduces to the correct result of Refs. 1 and 2.

If U is not a multiple of the unit matrix or does not transform as a singlet under finite symmetry group, then U has more than one distinct eigenvalue $U_a(x)$ and it does not commute with $D_\mu U$, $F_{\mu\nu}$, or other higher covariant derivatives. Therefore, with the exception of the first term in Eq. (11), it will not be possible to combine all factors of Δ into a single power to perform the momentum integration in a trivial manner. It is necessary to project out the eigenmodes $\Delta=\sum_a\Delta_a\mathcal{P}_a$ and then collect various factors of Δ_a for the momentum integration. It is important to point out that no other method including the proper-time heat kernel method in the present form can give the correct treatment for models with nondegenerate eigenmodes.[9]

For the purpose of isolating the divergent contribution one can safely ignore the noncommutivity of U with the covariant derivatives since the commutator $[\Delta,F_{\mu\nu}]=\Delta[F_{\mu\nu},U]\Delta$ contains an extra covergent factor. Therefore, the momentum integration can be performed easily. The logarthmically divergent term is given by

$$[i/(4\pi)^2](1/\epsilon)\mathrm{tr}[U^2-\tfrac{1}{6}F_{\mu\nu}^2],$$

which is identical to the well-known result of 't Hooft.[10]

We shall use the SO(N) linear σ model as an example to illustrate the method. The effective Lagrangean of this model is particularly interesting because it can be used to study the transition to the nonlinear σ model at $m_\sigma\to\infty$. The two-derivative terms have recently been calculated by Cheyette[2] using a slightly modified method of Ref. 1 and by Zuk.[5] However, the $m_\sigma\to\infty$ limit of these terms is trivial.

The SO(N) linear σ model Lagrangean is

$$\mathcal{L}(\Phi)=\tfrac{1}{2}(\partial_\mu\mathbf{\Phi})^2-\tfrac{1}{2}m^2\mathbf{\Phi}^2-(1/4!)\lambda(\mathbf{\Phi}^2)^2+\boldsymbol{\epsilon}\cdot\mathbf{\Phi},\quad(12)$$

where the last term is the symmetry-breaking term which would not contribute in this calculation. The effective action is given by[7]

$$\int d^4x\mathcal{L}_{\mathrm{eff}}(\mathbf{\Phi})=\int d^4x\mathcal{L}(\mathbf{\Phi})+\tfrac{1}{2}i\mathrm{Tr}\ln(-P^2+U(X)).\quad(13)$$

The matrix elements of U are most conveniently expressed in terms of the SO(N) spherical variables $\sigma=|\Phi|=(\mathbf{\Phi}^2)^{1/2}$ and $\boldsymbol{\phi}=\mathbf{\Phi}/|\Phi|$ such that $\boldsymbol{\phi}^2=1$,

$$\langle x|U_{ij}|y\rangle=[u_L(\sigma)P_{ij}^L+u_T(\sigma)P_{ij}^T]\delta^D(x-y),\quad(14)$$

where $\mathcal{P}_{ij}^L=\phi_i\phi_j$ and $\mathcal{P}_{ij}^T=\delta_{ij}-\phi_i\phi_j$ are the longitudinal and the transverse projection operators with multiplicities $n_L=\mathrm{Tr}\mathcal{P}^L=1$ and $n_T=\mathrm{Tr}\mathcal{P}^T=N-1$, respectively. The corresponding eigenvalues of U are

$$u_L(\sigma)=m^2+\tfrac{1}{2}\lambda\sigma^2,\quad u_T(\sigma)=m^2+\tfrac{1}{6}\lambda\sigma^2.\quad(15)$$

Similarly, we can express the propagation function as

$$\Delta=\Delta_L(\sigma)\mathcal{P}^L+\Delta_T(\sigma)\mathcal{P}^T\quad(16)$$

with

$$\Delta_L(\sigma)=1/[p^2+u_L(\sigma)],\quad\Delta_T(\sigma)=1/[p^2+u_T(\sigma)].$$

The effective Lagrangean can be obtained by substituting these explicit forms into Eq. (11) with the gauge fields set equal to zero and combining the result with Eq. (13),

$$\mathcal{L}_{\mathrm{eff}}=\mathcal{L}(\Phi)+\mathcal{L}_S(u_L)+(N-1)\mathcal{L}_S(u_T)+L_M,\quad(17)$$

where $\mathcal{L}(\Phi)$ is the Lagrangean in Eq. (12) and can be expressed in terms of the new variables σ and $\boldsymbol{\phi}$,

$$\mathcal{L}(\Phi)=\tfrac{1}{2}\sigma^2(\partial_\mu\boldsymbol{\phi})^2+\tfrac{1}{2}(\partial_\mu\sigma)^2-\tfrac{1}{2}m^2\sigma^2-(1/4!)\lambda\sigma^4+\boldsymbol{\epsilon}\cdot\boldsymbol{\phi}\sigma.\quad(18)$$

$\mathcal{L}_S(u)$ is the one-loop correction for the neutral scalar field as calculated in Refs. 1 and 2. In four dimensions, it is given by

$$\mathcal{L}_S(u)=\frac{1}{4(4\pi)^2}\left\{u^2\left[\frac{2}{\epsilon}-\ln\frac{u}{m^2}\right]+\tfrac{1}{6}u^{-1}(\partial u)^2+\tfrac{1}{60}u^{-2}(\partial^2 u)^2-\tfrac{1}{45}u^{-3}(\partial u)^2(\partial^2 u)+\tfrac{1}{120}u^{-4}(\partial u)^4\right\}. \tag{19}$$

$\mathcal{L}_M$ represents the contribution from two distinct eigenvalues u_L and u_T propagating in the same loop and is finite in four dimensions,

$$\begin{aligned}\mathcal{L}_M=(\tfrac{1}{3}\lambda)^2\int\frac{d^Dp_E}{(2\pi)^D}\Bigg\{&\frac{1}{D}p_E^2\Delta_L^2\Delta_T^2\sigma^4(\partial\boldsymbol{\phi})^2\\ &+\frac{2}{D(D+2)}p_E^4\Big[\Delta_L^2\Delta_T^2(3\Delta_L^2-\Delta_T^2)\sigma^4(\partial_\mu\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi})^2+4\Delta_L^2\Delta_T^4\sigma^4(\partial_\mu\boldsymbol{\phi}\cdot\partial_\nu\boldsymbol{\phi})^2\\ &+2\Delta_T^3\Delta_L^3\sigma^4(\partial^2\boldsymbol{\phi})^2+2\Delta_L\Delta_T(\Delta_T-3\Delta_L)(\Delta_T^3-3\Delta_L^3)\partial_\alpha\sigma^2\partial_\beta\sigma^2(\partial^\alpha\boldsymbol{\phi}\cdot\partial^\beta\boldsymbol{\phi})\\ &-4\Delta_L^2\Delta_T^2(\Delta_T^2-3\Delta_L^2)\sigma^2\partial_\mu\sigma^2(\partial^\mu\boldsymbol{\phi}\cdot\partial^2\boldsymbol{\phi})+2\Delta_L\Delta_T(\Delta_T^4-3\Delta_L^4)\sigma^2\partial^2\sigma^2(\partial\boldsymbol{\phi})^2\\ &-2\times\tfrac{1}{3}\lambda\Delta_L\Delta_T(\Delta_T^5-9\Delta_L^5)\sigma^2(\partial\sigma^2)^2(\partial\boldsymbol{\phi})^2\\ &-\tfrac{1}{2}(\tfrac{1}{3}\lambda)^2\Delta_L^2\Delta_T^2(9\Delta_L^4-3\Delta_L^2\Delta_T^2+\Delta_T^4)\sigma^4(\partial\sigma^2)^2(\partial\boldsymbol{\phi})^2\Big]\Bigg\}.\end{aligned} \tag{20}$$

The momentum integrals are of the type

$$\int\frac{d^Dp}{(2\pi)^D}p_E^{2S}\frac{1}{(p_E^2+u_L)^m(p_E^2+u_T)^n},$$

which can easily be integrated.[5] However, the result will not be very illuminating. It is more convenient to study the $m_\sigma\to\infty$ limit in the present form.

The present result for the two-derivative terms is in agreement with those calculated by Cheyette.[2] The four-derivative terms are completely new. It will be extremely difficult, if not impossible, to calculate them by any other method. I have also applied Eq. (11) to calculate the effective-action expansion for the nonlinear SO(N) σ model. The comparison between the linear and the nonlinear σ model will be reported elsewhere.[11]

Although I have directed the application of this method to the effective-action expansion, the general approach can be used for a wide range of interesting problems such as the operator-product expansion, anomalies, and other derivative expansions. One can also use Eq. (3) to develop a hybrid derivative expansion for a subset of background fields for the calculation of processes involving soft pions or photons.

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