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Non-Abelian Adiabatic Phases and the Fractional Quantum Hall Effect

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An explicit realization of U(n) quantum holonomy is presented in the context of nonrelativistic $(2+1)$-dimensional electrodynamics and used to construct the adiabatic effective action. It is shown that the U(1) subgroup of the U(n) holonomy gives rise to an effective Chern-Simons topological mass term and its relationship with the fractional quantum Hall effect is discussed.

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The quantum adiabatic phase[1,2] has recently emerged as a universal element in the topological analysis of various physical problems. It can be understood as an Aharonov-Bohm effect in the adiabatic transport of wave functions on the parameter space of a quantum mechanical system and has already seen numerous applications, particularly to the modification of semiclassical quantization rules,[2-4] fractional statistics of quasiparticles or vortices in two dimensions,[5-7] and the Schrödinger picture of chiral gauge anomalies.[8] Furthermore, some of its predicted interference phenomena[9] have recently found spectacular agreement with experiment.[10]

However, non-Abelian quantum holonomy,[2,11] with the notable exception of diatomic molecular systems, has encountered fewer applications. It requires the nongeneric scenario of a multiply degenerate quantum state which remains so over a finite subset of the parameter space of the Hamiltonian and, furthermore, that in this parameter subspace, no other energy levels cross those of the degenerate subset. In this Letter we shall construct an explicit example of this situation in the context of $(2+1)$-dimensional electrodynamics. We shall also derive an adiabatic effective action and show how the adiabatic phase is related to the Hall conductance of a fractionally filled degenerate energy band and to charges and statistics of magnetic flux tubes.

Recent phenomenological investigations[12] of the fractionally quantized Hall effect suggest dynamics derivable from an effective action with a Chern-Simons term.[13] Here, we shall provide a theoretical framework for this approach and demonstrate the appearance of topological photon mass.[14]

We consider a system of nonrelativistic electrons confined to a two-dimensional space with gyromagnetic ratio 2 and dynamics governed by the Pauli Hamiltonian

$$h_{\mathrm{P}}=\left(\frac{1}{2M}\right)\left(-i\hbar\nabla-\frac{e}{c}\mathbf{A}\right)^2+\left(\frac{e\hbar}{2Mc}\right)\beta B(x), \qquad (1)$$

where $B(x)=\nabla\times\mathbf{A}(x)$ and β is the spin matrix. (Both spin and magnetic field are scalars in two dimensions.)[15] We shall begin by second quantizing the fermions in an external static gauge field with $A_0=0$ and examining quantum holonomy of the fermionic ground state on the space of background fields.

Since (1) can be written as the square of the Dirac Hamiltonian,

$$h_{\rm P} = (2M)^{-1} h_{\rm D}^2,$$

where

$$h_{\rm D} = i\hbar \boldsymbol{\alpha}\cdot\nabla + (e/c)\boldsymbol{\alpha}\cdot\mathbf{A}, \tag{2}$$

they share the same eigenfunctions. We implement an infrared regularization by considering the operators (1) and (2) on a large two-sphere S^2. Then their spectra are discrete and the dimensions of their zero eigenspaces are given by the Atiyah-Singer index theorem[16]: $h_{\rm D}$ can be written in the form

$$h_{\rm D} = \begin{pmatrix} 0 & D \\ D^+ & 0 \end{pmatrix}, \tag{3}$$

and the index theorem yields

$$\dim\ker D - \dim\ker D^+ = \phi = \left(\frac{e}{2\pi\hbar c}\right)\int_{S^2} B(x). \tag{4}$$

Furthermore, $h_{\rm D}$ has a vanishing theorem on S^2—either $\dim\ker D = 0$ or $\dim\ker D^+ = 0$. This implies

$$\dim\ker h_{\rm D} = \dim\ker h_{\rm P} = |\phi|, \tag{5}$$

and leads to the desired scenario—$h_{\rm P}$ has a $|\phi|$-fold degenerate ground state which remains distinct from other components of the spectrum for all external fields $A(x)$ which support the same magnetic flux.[17]

If Ψ_α are the eigenfunctions of $h_{\rm P}$, $h_{\rm P}\Psi_\alpha = \omega_\alpha \Psi_\alpha$, the electron field is quantized as $\Psi(x) = \sum_\alpha a_\alpha \Psi_\alpha(x)$ where $\{a_\alpha, a_\beta^\dagger\} = \delta_{\alpha\beta}$. The ground state of the second quantized Hamiltonian, $H = \int_{S^2} \psi^\dagger(x) h_{\rm P} \psi(x)$, is $2^{|\phi|}$-fold degenerate—each zero mode being either filled or empty. However, we shall consider ground states which are also eigenstates of the electric charge and therefore have a fixed fractional filling f [f = (number of occupied zero modes)/$|\phi|$] and degeneracy $D = \binom{|\phi|}{f|\phi|}$.

We label these ground states by $|\mu;A\rangle$, $\mu = 1,\ldots, D$. They are defined by $a_\alpha|\mu;A\rangle = 0$ if $\omega_\alpha > 0$; $a_\alpha^\dagger|\mu;A\rangle = 0$ for a subset labeled by μ $\{\omega_\alpha = 0, \alpha\in\mu\}$ of $f|\phi|$ of the $|\phi|$ zero modes and $a_\beta|\mu;A\rangle = 0$ for the remaining $(1-f)|\phi|$ with $\omega_\beta = 0$, $\beta \in \mu$.

We consider the problem of adiabatic transport of a ground-state wave function around a continuous closed loop on the space of gauge-field configurations $A(x)$ with flux ϕ. We assume that the evolution is governed by the time-dependent Schrödinger equation

$$[i\partial/\partial t - H(A^t)]\psi(t,A^t) = 0, \quad 0 \le t \le T, \tag{6}$$

with the boundary condition $\psi(0) = |\mu;A^0\rangle$ and $A^T = A^0$. Since $H(A^t)|\mu;A^t\rangle = 0$ for $0\le t\le T$, we have the adiabatic solution

$$\psi(T,A^T) = \mathrm{P}\exp i\oint \mathcal{A}\, |\mu;A^T\rangle, \tag{7}$$

with the $U(D)$ functional connection

$$\mathcal{A}_{\mu\nu}(t) = i\langle\mu;A^t|\partial/\partial t|\nu;A^t\rangle. \tag{8}$$

The functional curvature is obtained by considering the two-parameter family $A(x,\sigma^a)$ $(a = 1,2)$ and the covariant curl of $\mathcal{A}$,

$$F_{\mu\nu} = (\delta\mathcal{A} + [\mathcal{A},\mathcal{A}])_{\mu\nu} = -i\epsilon^{ab}\sum_k{}'\left\langle\mu;A^\sigma\left|\frac{\partial}{\partial\sigma^a}\right|k;A^\sigma\right\rangle\left\langle k;A^\sigma\left|\frac{\partial}{\partial\sigma^b}\right|\nu;A^\sigma\right\rangle, \tag{9}$$

where the summation excludes intermediate states in the set of ground states degenerate in energy and charge with μ and ν. Since the operators $\partial/\partial\sigma^a$ conserve electric charge, this summation is further restricted to include only excited states. Therefore, we can use the formula

$$\left\langle\mu;A^\sigma\left|\frac{\partial}{\partial\sigma^a}\right|k;A^\sigma\right\rangle = \frac{1}{E_k}\left\langle\mu;A^\sigma\left|\frac{\partial H}{\partial\sigma^a}\right|k;A^\sigma\right\rangle \tag{10}$$

to evaluate (9).

This construction has an intrinsic functional gauge structure: Under an A-dependent change of basis in the zero eigenspace of H, $|\mu;A\rangle \to \Lambda_{\mu\nu}[A]|\nu;A\rangle$, where Λ is a $U(D)$ matrix, $\Lambda^+\Lambda = \Lambda\Lambda^+ = 1$, $\mathcal{A}$ and F transform like $\mathcal{A}\to\Lambda^{-1}(\delta+\mathcal{A})\Lambda$, $F\Lambda^{-1}\to F\Lambda$. The phase in (7) roman transforms covariantly and $\mathrm{tr}\,\mathrm{P}\exp(i\oint\mathcal{A})$ is invariant.

We shall now consider the gauge field effective action obtained by elimination of fermionic variables from the path-integral representation of the partition function. The propagator

$$K_{\mu\nu}[A] = \left\langle\mu;A^\beta\left|T\exp - \int_0^\beta dt\, H[A^t]\right|\nu;A^0\right\rangle, \tag{11}$$

with $A^\beta = A^0$, can be evaluated for large β in the adiabatic approximation by standard methods,

$$K_{\mu\nu}[A] = [\mathrm{P}\exp(i\oint\mathcal{A})]_{\mu\nu} + \ldots, \tag{12}$$

where from Eq. (8), $\mathcal{A}$ is of first order in $\dot{A}(x)$. Indeed, $K_{\mu\nu}[A]$ must transform covariantly under a change of basis, $K\to\Lambda^{-1}K\Lambda$, and therefore one would expect it to contain the path-ordered phase. For the Abelian case, this has been discussed in Ref. 4.

The effective action contains

$$\ln \operatorname{tr}(T\exp-\int_0^\beta H[A^t]\,dt)=\ln\operatorname{tr}\{\mathrm{P}\exp[i\oint_0^\beta \mathcal{A}(t)\,dt]\}+\ldots=\ln D+\left(\frac{i}{D}\right)\oint_0^\beta dt\,\operatorname{tr}\mathcal{A}(t)+\ldots, \tag{13}$$

where the correction terms are at least of second order in time derivatives of $A(x,t)$.[18]

Only the U(1) subgroup of the U(D) holonomy contributes in the adiabatic limit. The non-Abelian nature of A appears in higher orders. To obtain the gauge-invariant effective action we must add the Lagrange-multiplier field $A_a(x,t)$ with the charge density of the fermionic states. Their charge is $ef|\phi|$ and therefore their charge density, to leading order in derivatives, is[19]

$$\rho(x)=(e^2f/2\pi\hbar c)\operatorname{sgn}|\phi|B(x)+\ldots. \tag{14}$$

Thus, we obtain

$$S_{\mathrm{eff}}[A]=\ln D+\left(\frac{i}{D}\right)\oint_0^\beta d\tau\,\operatorname{tr}\mathcal{A}(\tau)+i\oint_0^\beta d\tau\int d^2x\left(\frac{e^2f}{2\pi\hbar c}\right)\operatorname{sgn}(\phi)A_0(x)B(x)+\ldots. \tag{15}$$

It remains to compute $\operatorname{tr}\mathcal{A}$. Gauge invariance of (15) would require that $\operatorname{tr}\mathcal{A}$ contains the term

$$\int d^2x\,(e^2f/4\pi\hbar c)\operatorname{sgn}(\phi)\epsilon^{ij}A_i(x,t)A_j(x,t).$$

This will be verified by explicit calculation. To analyze the Hall effect[20] it is sufficient to evaluate F in a constant external magnetic field on the plane R^2. With the gauge $A_{\mathrm{ext}}=(0,B_{\mathrm{ext}}x_1)$, and by use of Eqs. (9) and (10), it is straightforward to show that to leading order in $A(x)$,

$$\operatorname{tr}F=\int d^2x\,d^2y\,\epsilon^{ab}\frac{\partial A_i(x)}{\partial\sigma^\beta}\,\frac{\partial A_j(y)}{\partial\sigma^b}\operatorname{tr}F^{ij}(x,y,B^{\mathrm{ext}})+\ldots, \tag{16a}$$

$$\frac{1}{D}\operatorname{tr}F^{ij}(x,y)=\epsilon^{ij}\left(\frac{e^2f}{\hbar c}\right)\operatorname{sgn}(eB_{\mathrm{ext}})\sum_n\int\frac{dk\,dk'}{4\pi^2}\,\frac{\exp[-i(k-k')(x_2-y_2)]}{|eB_{\mathrm{ext}}|n}h_0\left(y_1-\frac{k}{|eB_{\mathrm{ext}}|}\right)$$
$$\times h_0\left(x_1-\frac{k}{|eB_{\mathrm{ext}}|}\right)h_n\left(y_1-\frac{k'}{|eB_{\mathrm{ext}}|}\right)h_n\left(x_1-\frac{k'}{|eB_{\mathrm{ext}}|}\right), \tag{16b}$$

where $h_n(x)$ are harmonic oscillator wave functions with frequency $|eB_{\mathrm{ext}}|$. Then, using Stokes's theorem we have

$$\operatorname{tr}\oint\mathcal{A}(\tau)\,d\tau=\frac{1}{2}\oint d\tau\int d^2x\,d^2y\,\dot{A}_i(x)\operatorname{tr}F^{ij}(x,yB_{\mathrm{ext}})A_j(y,\tau)+\ldots, \tag{17}$$

and upon substitution into Eq. (15) the local component of $\operatorname{tr}F^{ij}(x,y)$ yields the Hall conductivity tensor

$$\sigma^{ij}=\frac{1}{D}\int d^2x\,\operatorname{tr}F^{ij}(x,y)=\left(\frac{e^2f}{2\pi\hbar c}\right)\operatorname{sgn}(eB_{\mathrm{ext}})\epsilon^{ij}. \tag{18}$$

The form of the effective action further implies that a local approximation exhibits a topological mass term

$$S_{\mathrm{eff}}[A]=\ln D+\left(i\frac{e^2f}{2\pi\hbar c}\right)\operatorname{sgn}(eB_{\mathrm{ext}})\oint d\tau\,d^2x\,\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda+\ldots. \tag{19}$$

This expression [and implicitly Eq. (14)] yields fractional charges $q=ef\phi\operatorname{sgn}(eB_{\mathrm{ext}})$ for flux tubes with flux $e\phi/\hbar c$. Furthermore, an adiabatic argument in Ref. 6 indicates that it also describes fractional statistics—a wave functional for two tubes of flux $e\phi_1/\hbar c$ and $e\phi_2/\hbar c$ would change phase by $\pi f\phi_1\phi_2\operatorname{sgn}(eB_{\mathrm{ext}})$ upon interchange of their positions through a spatial rotation with angle π. These are precisely the charge and statistics of quasiparticles[5,6] in Laughlin's *Ansatz* for the ground-state wave function,[21] used in analysis of the fractionally quantized Hall effect. The further generation of topological photon mass provides the necessary conditions for the phenomenological approach of Ref. 12; the mass gap is required for the observed dissipationless current flow on the Hall conductivity plateaux. If current carriers are charged vortices, dissipation would necessitate excitation of plasmons which is suppressed by the mass gap.

In conclusion, we have found that adiabatic phases are related to the conductivity tensor conventionally obtained by linear-response theory. This suggests that our method could be used to evaluate the transport coefficients in other systems which manifest a similar nongeneric scenario for a degenerate quantum ground state.

We further observe that when f is an integer we obtain topological mass equal to the fermion induced mass in relativistic models,[14] and conductivity related to the TKNN integers[22] as first noted by Jackiw.[14]

Finally, the present adiabatic analysis obtains the kinematics of the Hall effect, the Hall conductance, and charges and statistics of quasiparticles. However, it does not address the apparently dynamical question of what determines the filling fraction f. Indeed, obtaining the desired[20] preferred values of f would require further dynamical analysis. One avenue would be to consider a background Wigner crystal of flux tubes. The topological mass term in (20) yields a Coulomb interaction and an effective action similar to the one derived for the recently proposed coherent-ring exchange theory of the quantized Hall effect.[23] Whether this approach can distinguish highly preferred values of f is a subject of ongoing research and a forthcoming publication.

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[15]In a realistic three-dimensional system $\boldsymbol{\mu}_{\text{ext}}\beta \cdot \mathbf{B}(x)$ is the Zeeman interaction. We would expect that in a very pure system, the electron's gyromagnetic ratio is exactly 2. However, if the effective gyromagnetic ratio is less than 2, the scenario we describe can still be achieved by tilting of the sample with respect to the external field—the Zeeman interaction is approximately independent of the direction of the external field whereas the electron dynamics depends only on its normal component.

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[17]Note that (4) and (5) imply that the magnetic flux is quantized. In fact, we can have magnetic flux on S^2 only if when embedded in a three-dimensional space the S^2 encloses magnetic poles. Quantization of ϕ is equivalent to the Dirac quantization condition.

[18]If $A(x,t)$ is periodic on $[0,\beta]$, then $\dot{A}(x,t) \sim \beta^{-1} A(x,t)$ and adiabaticity implies low temperature. We shall consider the regime where β^{-1} is much less than the gap between zero modes and first excited states of H.

[19]In the adiabatic limit, only ground states with charge $ef|\Phi|$ are included in intermediate states in the propagator (13). The charge densities of these states differ only by the higher derivative corrections to (17).

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