Length-Twist Parameters in Stiing Path Integrals

Eric D'Hoker and D. H. Phong

Departments of Physics and Mathematics, Columbia University, New York, New York 10027

(Received 20 December 1985)

An approach to multiloop string path integrals is presented which expresses them in terms of Green's functions on triply connected domains or "pants."

PACS numbers: 11.17.+y, 12.10.Gq

String theories are promising candidates for realistic quantum theories of matter and gravity. The type-I SO(32), type II, and heterotic strings are likely to be finite to one-loop order.[1] Recently progress toward the understanding of multiloop amplitudes has been made both in the light-cone and in the covariant formulations.[2,3] In particular, in Ref. 3 we have obtained the following formula for the Polyakov partition function for a closed bosonic string and an arbitrary number h of loops in the critical dimension $d=26$:

$$Z_h = \int_{\mathrm{Moduli}(h)} (\det \hat{P}_1^\dagger \hat{P}_1)^{1/2} \left(\frac{2\pi}{\int_M d^2\xi\, \hat{g}^{1/2}} \det{}' \Delta_{\hat{g}} \right)^{-13} d(\text{Weil-Petersson}). \tag{1}$$

Recall that the world sheet in this case is a closed surface M with h handles, and Moduli(h) is the space of conformal structures (conformal classes of metrics, distinct under reparametrizations) on M. Each element of Moduli(h) can be represented by a metric $\hat{g}$ of curvature -1 if $h \geqslant 2$, curvature 0 and area 1 if $h=1$. The tangent space to Moduli(h) at $\hat{g}$ can then be viewed as the space of quadratic differentials, which is of dimension 2 when $h=1$, and of dimension $6h-6$ when $h \geqslant 2$. The Weil-Petersson metric on Moduli(h) is obtained by pairing quadratic differentials and integrating over M through use of $\hat{g}$. The operators $\Delta_{\hat{g}}$ and $\hat{P}_1^\dagger \hat{P}_1$ are respectively the Laplacians on scalars and vectors. [For scattering amplitudes there are additional factors in (1) involving Green's functions.] The determinants of $\Delta_{\hat{g}}$ and $\hat{P}_1^\dagger \hat{P}_1$ have been expressed as special values of the Selberg zeta function and its derivative.[3]

In this Letter we shall present a different approach based on characterizations of conformal structures by length-twist or "Fenchel-Nielsen" parameters. Besides their simple geometric interpretation, these parameters have the advantage of yielding a simple formula for the Weil-Petersson Kähler form and measure,[4] and of being well adapted to the Feynman path-integral formalism. The formula we obtain this way represents Z_h (for $h \geqslant 2$; for $h=1$ it reproduces the classical formula for one-loop amplitude[5]) in terms of Green's functions on "pants," i.e., constant-negative-curvature surfaces with geodesic boundaries and the topology of a sphere with three holes cut out. The Green's functions on pants are likely to be intimately linked with an off-shell version of the full three-string vertex.

The basic building block of negative-curvature surfaces is the "pair of pants," which is topologically a sphere with three holes cut out. A conformal structure on the pants can be represented by a metric of curvature -1, with respect to which the three boundaries $\gamma_1, \gamma_2, \gamma_3$ are geodesics. Such a metric is in turn characterized by the lengths l_1, l_2, and l_3 of the three boundary geodesics. This can be seen by noting that the pants arise from gluing together two isometric hexagons, which are obtained by cutting the pants along three geodesics Σ_1, Σ_2, Σ_3 perpendicular to γ_2 and γ_3, γ_3 and γ_1, γ_1 and γ_2, respectively (see Fig. 1). Basic hyperbolic geometry shows that a hexagon with right angles and geodesic sides is completely specified by the lengths (which can be chosen arbitrarily) of

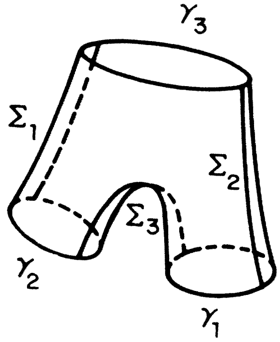

FIG. 1. Decomposition of pants into two isometric hexagons.

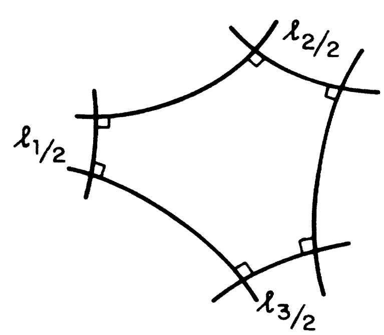

FIG. 2. Hexagons in hyperbolic geometry.

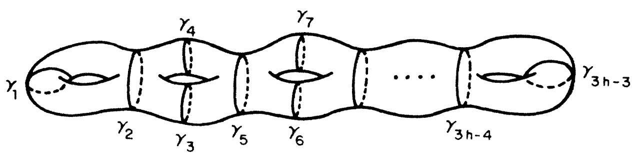

FIG. 3. Decomposition into pants.

three alternating sides, in particular $l_1/2$, $l_2/2$, and $l_3/2$. (See Fig. 2.)

Now a compact surface M with h (≥ 2) handles can be decomposed into $2h-2$ pairs of pants P_j, $1 \leq j \leq 2h-2$, by cutting M along $3h-3$ closed curves γ_i, $1 \leq i \leq 3h-3$ (see Fig. 3). To obtain a constant-curvature metric $\hat{g}$ on M, we begin by selecting constant-curvature metrics on the P_j's which can be glued together along the γ_i's. The metrics on two pairs of pants will extend across the gluing if it takes place along a geodesic of the same length. Thus we can take an arbitrary set of values $(l_i) \in R_+^{3h-3}$ for the lengths of the γ_i's and glue together the corresponding metrics in the P_j's. A natural way of gluing is to match the corners of the hexagons described above, but we can also glue after making a relative twist of an arbitrary angle $\phi_i \in R$ along γ_i (see Fig. 4). The constant-curvature metrics on M obtained this way represent precisely Teichmüller space T_h, i.e., conformal classes of metrics obtained under reparametrizations continuously deformable to the identity. To

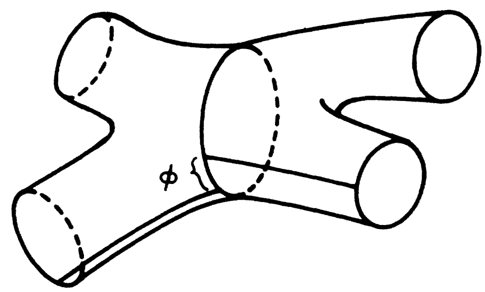

FIG. 4. A gluing with twist of an angle ϕ.

obtain Moduli(h), we still have to divide $T_h \simeq (R_+ \times R)^{3h-3}$ by the mapping class group Γ_h, which is the group of the disconnected diffeomorphisms of M. The mapping class group evidently contains the subgroup Z^{3h-3} generated by the integral twists. It is in general a complicated object, and it is an important question to determine its fundamental domain in $(R_+ \times R)^{3h-3}$.

In terms of the length-twist coordinates $(l_i, \phi_i) \in (R_+ \times R)^{3h-3}$, the Weil-Petersson Kähler form can be expressed as[4]

$$\omega_{\mathrm{WP}} = \sum_{i=1}^{3h-3} dl_i \wedge d\phi_i,$$

so that d(Weil-Petersson) reduces to $dl_i \cdots dl_{3h-3} \times d\phi_1 \cdots d\phi_{3h-3}$, and the partition function becomes

$$Z_h = \frac{(4h-4)^{13}}{|\Gamma_h/Z^{3h-3}|} \int_0^{2\pi} d\phi_1 \cdots d\phi_{3h-3} \int_0^{\infty} dl_1 \cdots dl_3 (\det \hat{P}_1^\dagger \hat{P}_1)^{1/2} (\det{}' \Delta_{\hat{g}})^{-13}. \tag{2}$$

Thus the main problem is now to compute the determinants in terms of (l_i, ϕ_i). In the following we shall reduce $\det{}'\Delta_{\hat{g}}$ to Gaussian integrals over objects involving only Green's functions on pants with zero Dirichlet boundary conditions. We expect a similar treatment to apply to $\det{}'\hat{P}_1^\dagger \hat{P}_1$ with the help of vector ghosts.

To calculate $\det{}'\Delta_{\hat{g}}$ we return to its path-integral expression,

$$\left[\frac{2\pi}{\int_M d^2\xi\, \hat{g}^{1/2}} \det{}'\Delta_{\hat{g}}\right]^{-1/2} = \int \mathscr{D}'X\, e^{-I_0[X,\hat{g}]}, \quad I_0[X,\hat{g}] = \frac{1}{2}\int_M d^2\xi \sqrt{\hat{g}}\, \hat{g}^{ab} \partial_a X\, \partial_b X.$$

In the functional measure $\mathscr{D}'X$, the zero mode is omitted and the measure $\mathscr{D}X$ is normalized by the usual requirement of ultralocality. Also, we shall always use zeta-function regularization of determinants, so that no infinite counterterms are needed.

The partition of the surface M introduced previously naturally divides the X integration into $2h-2$ functional integrations over pants. This is the two-dimensional analog of the fact that the Feynman path integral in quantum mechanics may be split into time segments. The full path integral is then obtained by a summation over the intermediate values of the dynamical variables.

If, for some quantum-mechanical system with classical action $S(q,\dot{q})$, we define

$$K[q_1,t_1;q_2,t_2] = \int_{q(t_1)=q_1,\, q(t_2)=q_2} \mathscr{D}q\, e^{-S(q,\dot{q})},$$

then we have for $t_1 < t_2 < t_3$,

$$K[q_1,t_1;q_3,t_3] = \int dq_2\, K[q_1,t_1;q_2,t_2] K[q_2,t_2;q_3,t_3]. \tag{3}$$

Returning to our integral over the surface M, the analog is to integrate over the values of X on the partitioning geodesics. However, some care is needed in precisely defining the boundary condition. In the one-dimensional case there is no ambiguity, while in the two-dimensional case X may perform a rigid twist corresponding to the ϕ_i's discussed earlier.

Thus we fix a parametrization $\gamma_i(\theta)$ for the geodesics γ_i and define, for given $l_i, l_j, l_k > 0$ and periodic functions $x_i(\theta), x_j(\theta), x_k(\theta)$,

$$F_{l_il_jl_k}[x_i,x_j,x_k] = \int_{X(\gamma_I(\theta))=x_I(\theta),\ I=i,j,k} \mathscr{D}X\, e^{-I_0[X,\hat{g}]}. \tag{4}$$

Here $\hat{g}$ is the metric of curvature -1 on the pants determined by the lengths l_i, l_j, l_k for $\gamma_i, \gamma_j, \gamma_k$. In analogy with (3) we have now the following formula[6] for the metric $\hat{g}$ on M corresponding to the lengths $l_1, \ldots, l_{3h-3}$ and twists $\phi_1, \ldots, \phi_{3h-3}$:

$$\left[\frac{1}{4(h-1)}\det{}'\Delta_{\hat{g}}\right]^{-1/2} = \int D'x_1\, Dx_2 \cdots Dx_{3h-3}\, F_{l_1l_1l_2}[x_1, x_1(\phi_1+\cdot), x_2]$$
$$\times F_{l_2l_3l_4}[x_2(\phi_2+\cdot), x_3, x_4] F_{l_3l_4l_5}[x_3(\phi_3+\cdot), x_4(\phi_4+\cdot), x_5]$$
$$\times \cdots F_{l_{3h-4}l_{3h-3}l_{3h-3}}[x_{3h-4}(\phi_{3h-4}+\cdot), x_{3h-3}, x_{3h-3}(\phi_{3h-3}+\cdot)]. \tag{5}$$

In the integral $D'x_1$ we delete the contribution from the zero mode.

To compute $F_{l_il_jl_k}[x_i,x_j,x_k]$ from (4), we note that $X(\xi)$ can be uniquely decomposed as $X(\xi) = \chi(\xi) + \phi(\xi)$, where $\phi(\gamma_I(\theta)) = 0$, and χ is the harmonic function on pants with $\chi(\gamma_I(\theta)) = x_I(\theta)$, $I = i,j,k$. Since $I_0[X,\hat{g}]$ becomes then

$$I_0[X,\hat{g}] = \frac{1}{2}\int_{\text{pants}} d^2\xi \sqrt{\hat{g}}\, \hat{g}^{ab}\, \partial_a\varphi\, \partial_b\varphi + \sum_{I=i,j,k} \frac{1}{2}\oint_{\gamma_I} d\hat{n}^a \chi\, \partial_a \chi,$$

and the functional measure $\mathscr{D}X$ reduces to $\mathscr{D}\phi$, we obtain

$$F_{l_il_jl_k}[x_i,x_j,x_k] = \left(\int \mathscr{D}\varphi\, e^{-I_0[\phi,g]}\right) \exp\left(-\frac{1}{2}\sum_{I=i,j,k}\oint_{\gamma_I} d\hat{n}^a \chi\, \partial_a\chi\right) = \psi(l_i,l_j,l_k) \exp\left(-\frac{1}{2}\sum_{I=i,j,k}\oint_{\gamma_I} d\hat{n}^a \chi\, \partial_a\chi\right), \tag{6}$$

where $\psi(l_i,l_j,l_k)^{-2}$ is just the determinant of the Laplacian on pants with zero Dirichlet conditions. We observe that $\psi(l_i,l_j,l_k)$ can be in principle obtained from the Green's function on pants, $G_{l_il_jl_k}(\xi,\xi')$. As for χ, it can be written explicitly as

$$\chi(\xi) = \sum_{I=i,j,k} \oint_{\gamma_I} x_I(\theta)(\partial/\partial n') G_{l_il_jl_k}(\xi, \gamma_I(\theta))\, d\theta.$$

Consequently,

$$\sum_{I=i,j,k} \frac{1}{2}\oint_{\gamma_I} d\hat{n}^a \chi\, \partial_a\chi = \sum_{I,J=i,j,k} \oint_I d\theta' \oint_J d\theta''\, x_I(\theta') x_J(\theta'') K_{l_il_jl_k}(\theta',\theta''),$$

where

$$K_{l_il_jl_k}(\theta',\theta'') = \sum_{I=i,j,k} \oint_{\gamma_I} d\theta (\partial/\partial n') G_{l_il_jl_k}(\theta,\theta')(\partial^2/\partial n''\,\partial n) G_{l_il_jl_k}(\theta,\theta''). \tag{7}$$

The formulas (7), (6), (5), and (2) together provide a systematic way of determining the partition function for any number of loops from a basic ingredient, which is the Green's function on pants. Such functions can be constructed out of Poincaré series or studied by direct analytic methods.[7] It would be valuable to obtain any detailed information on their dependence on the parameters l_i, l_j, l_k.

The approach just described for $h \geqslant 2$ applies with only slight changes to the case $h = 1$, and yields easily the familiar formulas for one loop. In this case the world sheet is a torus, which is partitioned by a single geodesic γ of length l, and twist angle ϕ. The surface that remains after this partition is just the cylinder, whose conformal structures are represented by flat metrics $\hat{g}$ parametrized by the single coordinate l. The F function for the cylinder now depends only on two boundary conditions $x_1(\theta)$ and $x_2(\theta)$, and the same reasoning as used previously leads to the following formulas:

$$(2\pi \det{}'\Delta)^{-1/2}_{\text{torus}} = \int D'x\, F_l[x, x(\phi+\cdot)],$$
$$F_l[x_1,x_2] = \psi(l)\exp\left(-\tfrac{1}{2}\oint_\gamma dn^a \chi\, \partial_a\chi\right). \tag{8}$$

Here $\psi^{-2}(l)$ is the determinant of the flat metric on the cylinder with zero boundary conditions, and χ is the harmonic function with boundary values χ_1 and χ_2.

If we parametrize the cylinder by $0 \leqslant \xi^1 \leqslant 1/l$ and $0 \leqslant \xi^2 \leqslant l$ and use a Fourier decomposition for x_1 and x_2,

$$x_{1(2)}(\xi^2) = \sum_{m=-\infty}^{\infty} c_m^{1(2)} e^{2\pi i m \xi^2/l},$$

we obtain by a straightforward computation

$$F_l[x_1,x_2] = \psi(l)\exp\left\{-\sum_{m=-\infty}^{\infty} \frac{m\pi}{\sinh(2\pi m/l^2)}[\cosh(2\pi m/l^2)(|c_m^1|^2+|c_m^2|^2) - (c_m^{1*}c_m^2 + c_m^1 c_m^{2*})]\right\}. \tag{9}$$

We note that this expression appeared previously in the study of a propagator for the second-quantized bosonic string.[8] Now a gluing after a twist ϕ means that $x_2(\theta) = x_1(\phi+\theta)$, so that $c_m^1 = c_m$ and $c_m^2 = c_m e^{2\pi i m\phi/l}$, and thus

$$F_l[x,x(\phi+\cdot)] = \psi(l)\exp\left\{-\sum_{m=1}^{\infty}\frac{4m\pi}{1-\exp(-4\pi m/l^2)}|c_m|^2\left|1-\exp\left[2\pi i m\left(\frac{\phi}{l}+il^{-2}\right)\right]\right|^2\right\}.$$

The x functional integral is then easily performed, and we find

$$\det{}'\Delta = \psi(l)^{-2}\prod_{m=1}^{\infty}\left(\frac{4\pi m/l}{1-\exp(-4\pi m/l^2)}\right)^2\prod_{m=1}^{\infty}\left|1-\exp\left[2\pi i m\left(\frac{\phi}{l}+il^{-2}\right)\right]\right|^4 l^{-1}.$$

Next $\psi(l)$ can be computed since the eigenvalues of the Laplacian on the cylinder with zero Dirichlet conditions are just

$$l^2\pi^2 n^2 + 4\pi^2 m^2/l^2, \quad 1 \leqslant n \leqslant \infty, \quad -\infty \leqslant m \leqslant \infty.$$

A zeta-function regularization and the Watson-Sommerfeld transform yield (see Ref. 5)

$$\psi(l) = \frac{l}{\sqrt{2}} e^{\pi/6l^2}\prod_{n=1}^{\infty}(1-e^{-4\pi n/l^2})^{-1},$$

and hence

$$\det{}'\Delta = \frac{1}{l^2}e^{-\pi/3l^2}\prod_{m=1}^{\infty}\left|1-\exp\left[2\pi i m\left(\frac{\phi}{l}+il^{-2}\right)\right]\right|^4.$$

For the torus $(\det{}' P_1^\dagger P_1)^{1/2} = (\det{}'\Delta)/2$, the mapping class group is SL(2,Z), and the Weil-Petersson Kähler form is $4\, dl \Delta\, d\phi$ (the factor of 4 comes from our normalization of area). The final formula for Z_1 thus becomes

$$Z_1 = \frac{4}{|\mathrm{SL}(2,Z)|}\int_0^\infty dl\int_{-\infty}^{\infty} d\phi \frac{1}{4\pi}\left(\frac{l^2}{2\pi}\right)^{12} e^{4\pi/l^2}\left\{\prod_{m=1}^{\infty}\left|1-\exp\left[2\pi i m\left(\frac{\phi}{l}+il^{-2}\right)\right]\right|\right\}^{-48}. \tag{10}$$

Changing variables to $\tau_1 = \phi l^{-1}$, $\tau_2 = 1/l^2$, we recognize $2\, dl\, d\phi$ as the SL(2,Z) invariant measure $d\tau_1\, d\tau_2/\tau_2^2$, and Z_1 as given by the well-known formula obtained in Refs. 3 and 5.

We have benefitted from conversations with E. Martinec, S. Plotnick, and J. Polchinski. One of us (E.D.) wishes to thank the Aspen Center for Physics for its hospitality. This work was supported in part by the U.S. Department of Energy and by the National Science Foundation under Grant No. DMS 84-02710.

[1]J. H. Schwarz, Phys. Rep. **89**, 223 (1982); M. B. Green, Surv. High Energy Phys. **3**, 127 (1983); M. B. Green and J. H. Schwarz, Phys. Lett. **151B**, 21 (1985); D. J. Gross, J. A. Harvey, E. Martinec, and R. Rohm, Phys. Rev. Lett. **54**, 502 (1985), and to be published; L. Clavelli, University of Alabama Report No. 85-0742 (to be published); P. Frampton, P. Moxhay, and J. Ng, Phys. Rev. Lett. **55**, 2107 (1985).

[2]O. Alvarez, Nucl. Phys. **B216**, 125 (1983); G. Moore and P. Nelson, Harvard University Report No. HUTP-85/A057 (to be published); S. Mandelstam, University of California at Berkeley Report No. UCB-PTH-85/47 (to be published); G. Gilbert, University of Texas Report No. UTTG-23-85 (to be published).

[3]E. D'Hoker and D. H. Phong, Columbia University Report No. CU-TP-323 (to be published).

[4]S. Wolpert, Ann. Math. **117**, 207 (1983).

[5]J. Polchinski, University of Texas Report No. UTTG-13-85 (to be published); J. Shapiro, Phys. Rev. D **5**, 1945 (1972).

[6]We denote functional integrals over functions on M by $\mathscr{D}X$, and those over functions on the circle by Dx.

[7]M. Schiffer and D. C. Spencer, *Functionals of Finite Riemann Surfaces* (Princeton Univ. Press, Princeton, N.J., 1954).

[8]A. Cohen, G. Moore, P. Nelson, and J. Polchinski, Harvard University Report No. HUTP-85/A 058 and University of Texas Report No. UTTG-16-85 (to be published).