# Classical Limit of Bethe-*Ansatz* Thermodynamics for the Sine-Gordon System

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We use the quantum Bethe-*Ansatz* method to compute the free energy of the classical sine-Gordon system. Previous attempts to do this have failed because the number of coupled integral equations to be solved to find the free-energy diverges in the classical limit. We present a transformation, extending a method of Maki, which reduces this divergent set to only two coupled equations, for the densities of solitons and *anharmonic* phonons. These equations can be solved iteratively in the temperature $t$ and the soliton density $e^{-1/t}$ to give a double series for the free energy. This series coincides to high order with classical transfer-matrix results.

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There has been much recent interest in the thermodynamic properties of soliton-bearing systems, and, in particular, a great deal of effort has gone into analyzing the $(1+1)$-dimensional sine-Gordon model in both its classical and quantum formulations.[1,2] The sine-Gordon Hamiltonian has been used to describe a wide variety of physical systems, and its integrable nature means that a more complete mathematical analysis is possible than for general nonlinear systems. For the *classical* case,[1] the transfer-matrix method has been used to find the free energy as a double series in $t$ (temperature in soliton mass units) and $e^{-1/t}$ to high order. In the semiclassical regime, the gas phenomenologies[2] have been very helpful in developing a physical understanding of the system as an interacting gas of solitons, breathers, and/or phonons. However, in the classical limit these methods yield only the lowest-order terms in the free-energy series, partly because the well-known phonon-breather double-counting difficulties have not been resolved in an entirely satisfactory fashion. For the *quantum* case, the Bethe *Ansatz* is a mathematically well-defined method for evaluating the free energy by solving a set of coupled integral equations for the densities of the soliton and breather excitations.[3,4] These equations can be solved numerically[4] to any desired accuracy—except, unfortunately, near the classical limit where the number of allowed breathers, and hence of coupled equations, diverges. (The energy difference between allowed breathers is $\sim h/\tau$, where $\tau$ is the period of the classical soliton-anitsoliton bound state, and the classical limit is $h \rightarrow 0$.) Thus it has not previously been possible to take the Bethe-*Ansatz* analysis to the classical limit and link up with the transfer-matrix results.

In the present paper, we demonstrate that the divergent number of coupled integral equations that we get on taking the classical limit of the Bethe-*Ansatz* thermodynamic analysis can be transformed in that limit into two coupled integral equations for the densities of solitons and anharmonic phonons. In other words, the sets of densities of breathers can be summed to give an equivalent anharmonic-phonon density. This pair of coupled equations can then be solved iteratively in temperature $t$ and $e^{-1/t}$ to reproduce the classical transfer-matrix results to high order [see Eq. (21)]. That is to say, we have linked up the quantum Bethe-*Ansatz* results with the classical transfer-matrix results. Indeed, the Bethe *Ansatz* proves to be just as efficient as the classical methods for computing the free energy of the classical system.

The basic idea for the reduction of the breather densities to a single equivalent phonon density was first used by one of us[5] in the simple limit given by letting the soliton mass go to infinity while keeping the phonon mass and temperature fixed. In this case, the system becomes a free massive phonon gas, and the breathers are zero-binding-energy "bound states" of phonons. The Bethe-*Ansatz* thermodynamic analysis still generates a large set of coupled integral equations for these breather densities, but, not surprisingly, these equations can be solved analytically to give the free phonon thermodynamics. The next step came when Maki[6] applied this same breather-to-phonon transformation in the nontrivial classical limit where soliton mass and temperature remain finite, but the phonon mass goes to zero. In this limit, the breather binding energy (regarded as a bound state of phonons) is no longer zero. Thus replacing the breathers by a free phonon gas necessarily neglects anharmonic contributions to the free energy and can only give leading terms in the free-energy series. Here we show that the anharmonic coupling can in fact be incorporated into the same mathematical framework, by retaining the $1/n$ correction to the Bethe-*Ansatz* breather-breather phase shifts in the classical limit. Thus the many breather densities can be replaced by a single density of anharmonic phonons, giving correctly the anharmonic terms in the free-energy series in the classical limit.

We first write down the Bethe-*Ansatz* (BA) thermodynamic equations for the functions $\eta_j(\beta)$ (ratios of local densities of empty states to filled states). For the sine-Gordon (SG) coupling parameter $\mu=(1-1/n)\pi$, where $n$ is an integer and $n\to\infty$ is the classical limit,[4]

$$\ln\eta_j(\beta)=\frac{M_j}{T}\cosh\left(\frac{\pi\beta}{2\mu}\right)+\frac{1}{2\pi}\sum_{k=1}^{n-1}\int_{-\infty}^{\infty}d\beta'\frac{d}{d\beta'}\theta_{jk}(\beta'-\beta)\ln[1+\eta_k^{-1}(\beta')], \tag{1}$$

where

$$M_j=2M\sin[j\pi/2(n-1)] \tag{2}$$

is the mass of the $j$th breather ($j\neq n-1$), and $M_{n-1}=2M$, with $M$ the soliton mass. The breather-breather phase shifts are

$$\theta_{jk}(\beta)=\theta(\beta,j+k)+\theta(\beta,|j-k|)+2\sum_{m=1}^{\min(j,k)-1}\theta(\beta,j+k-2m), \tag{3}$$

with

$$\theta(\beta,j)=-i\ln\left(\frac{\sinh(\pi\beta/2\mu)-i\sin[j\pi/2(n-1)]}{\sinh(\pi\beta/2\mu)+i\sin[j\pi/2(n-1)]}\right). \tag{4}$$

To find the free energy $F$, Eqs. (1) must be solved for the $\eta_j(\beta)$ which are then substituted into

$$F=-\frac{T}{2\pi}\sum_{j=1}^{n-1}\int_{-\infty}^{\infty}d\left(\frac{\pi\beta}{2\mu}\right)M_j\cosh\left(\frac{\pi\beta}{2\mu}\right)\ln[1+\eta_j^{-1}(\beta)]. \tag{5}$$

The important analytic structure which makes it possible to reduce Eqs. (1) to just two coupled equations is most easily seen in the rather trivial limit defined by allowing the soliton mass $M$ to become infinite as the coupling $\mu\to\pi$, keeping the BA phonon (lowest breather) mass $m=\pi M/n\approx M_1$ finite, and the temperature finite.[5] It is easy to see from (2) and (3) that in this limit the mass spectrum is uniformly spaced,

$$M_j\approx M\pi j/n=jm, \tag{6}$$

and the phase-shift derivatives become delta functions:

$$\theta_{jk}'(\beta)=4\pi\min(j,k)\delta(\beta),\quad j\neq k;\quad \theta_{jj}'(\beta)=2\pi(2j-1)\delta(\beta). \tag{7}$$

The thermodynamic equations in this limit can be solved analytically to give[5]

$$1+\eta_j^{-1}(\beta)=\frac{\sinh^2[(j+1)C/2T]}{\sinh(jC/2T)\sinh[(j+2)C/2T]}, \tag{8}$$

where $C=m\cosh(\pi\beta/2\mu)$. Using the identity

$$\sum_k k\ln[1+\eta_k^{-1}(\beta)]=-\ln\left\{1-\exp\left[-\frac{m}{T}\cosh\left(\frac{\pi\beta}{2\mu}\right)\right]\right\}, \tag{9}$$

we find, with $\alpha=\pi\beta/2\mu$, that

$$F=\frac{mT}{2\pi}\int_{-\infty}^{\infty}d\alpha\cosh\alpha\ln\left[1-\exp\left(-\frac{m}{T}\cosh\alpha\right)\right], \tag{10}$$

which is just the free energy of a noninteracting phonon gas.

This is not a surprising result—in this limit the sine-Gordon equation becomes the Klein-Gordon (KG) equation, and these are the KG phonons. An important point is that the KG phonon is a different physical object from the BA phonon (lowest breather) in the thermodynamic system. The KG phonons do not phase shift each other in this limit, and many of them can occupy the same state. The BA phonons do phase shift each other, and also the breathers (which here are zero-binding-energy bound states of BA phonons, and are not present in the KG picture). The multiple occupancy of a single state by KG phonons is represented in the BA picture by sets of breathers containing particles at the same rapidity. Thus in this infinite-$n$, infinite-soliton-mass limit, the KG phonons are equilvalent to the *whole ladder* of BA breathers.

Recently Maki[6] reanalyzed Eqs. (1) in the $n\to\infty$ limit, keeping the soliton mass and the temperature finite, so that the phonon mass diminishes as $1/n$.

This is a more appropriate classical limit than that discussed above for representing a physical sine-Gordon system. He used the breather spectrum and phase shifts (6) and (7)—that is to say, he replaced the ladder of breathers by harmonic KG phonons. However, in contrast to the limit discussed above, the soliton now has finite mass and must be included in the BA thermodynamic equations. The soliton-soliton and soliton-breather phase shifts for large $n$ take the forms

$$\theta'_{ss}(\beta)=\frac{n}{2\pi}\ln\left(\frac{\cosh\alpha+1}{\cosh\alpha-1}\right), \tag{11}$$

$$\theta'_{sj}(\beta)=j\,\mathrm{sech}\alpha. \tag{12}$$

The important point here is that the soliton–$j$th-breather phase shift is proportional to $j$, which is consistent with our regarding the $j$th breather as simply $j$ KG phonons. Thus in Maki's approximation the system has two kinds of excitations—solitons and KG (harmonic) phonons—and Eqs. (1) reduce to a set of two coupled equations for functions $w(\alpha)$ and $\eta_s(\alpha)$ related to the phonon and soliton densities. Using the formula analogous to Eq. (9),

$$\sum_{k=1}^{n-2}k\ln[1+\eta_k^{-1}(\alpha)]=-\ln[1-e^{-w(\alpha)}]+O(e^{-\pi/t}), \tag{13}$$

Maki shows that (1) becomes

$$w(\alpha)=\frac{m}{T}\cosh\alpha+\frac{4}{2\pi}\int_{-\infty}^{\infty}d\alpha'\frac{1}{\cosh(\alpha'-\alpha)}\ln[1+\eta_s^{-1}(\alpha')], \tag{14}$$

$$\ln\eta_s(\alpha)=\frac{M}{T}\cosh\alpha-\frac{2}{2\pi}\int_{-\infty}^{\infty}d\alpha'\frac{1}{\cosh(\alpha'-\alpha)}\ln[1-e^{-w(\alpha')}]+\frac{n}{\pi^2}\int_{-\infty}^{\infty}d\alpha'\ln\left(\frac{\cosh(\alpha'-\alpha)+1}{\cosh(\alpha'-\alpha)-1}\right)\ln[1+\eta_s^{-1}(\alpha')]. \tag{15}$$

These equations differ from the standard BA form because the KG phonons are bosonic, not quasifermionic. Solving (14) and (15) by iteration in $e^{-1/t}$, where $t=T/M$ is the reduced temperature, and then using (5) gives the leading one- and two-soliton contributions to the free energy correctly, but higher-order terms depend on the neglected phonon-phonon interactions.

In the present paper, we improve on Eqs. (14) and (15) by replacing the harmonic KG phonon with anharmonic SG phonons. We work within the BA formalism, taking the leading correction (in $1/n$) to the delta-function phase shifts (7) which led to the harmonic KG phonons. In the limit $n\to\infty$, this appears to include all finite soliton and anharmonic-phonon contributions to the thermodynamic equations.

It is convenient to work in $x$ space, where $x$ is the Fourier transform variable of the rapidity $\alpha$. For large $n$, from (4), for $j\neq0$,

$$\theta'(x,j)=\left(\frac{\pi}{2}\right)^{1/2}\frac{\cosh[(\pi/2-j\pi/2n)x]}{\cosh[(\pi/2)x]}=\left(\frac{\pi}{2}\right)^{1/2}\left[1-\frac{j\pi}{2n}x\tanh\left(\frac{\pi x}{2}\right)\right]+O\left(\frac{1}{n^2}\right). \tag{16}$$

From (3) it is easy to check that the $1/n$ term above gives an extra phase shift between the $j$th and $k$th breathers proportional to $jk$, and therefore can be interpreted as a phase shifting between $j$ SG phonons and $k$ SG phonons. Thus the phonon-soliton picture is not spoiled, and the total effect of our including this interaction is to add the following to the right-hand side of (14):

$$(1/2n)\int_{-\infty}^{\infty}dx\cos(\alpha x)x\tanh(\pi x/2)\int_{-\infty}^{\infty}d\alpha'\cos(\alpha' x)\ln[1-e^{-w(\alpha')}], \tag{17}$$

representing the phonon-phonon interaction.

To investigate the classical limit (in which the phonon mass goes to zero), it is convenient to rescale the density functions, following Maki, and define

$$w(\alpha)=(m/T)u(\alpha),\quad \eta_s^{-1}(\alpha)=(m/T)g(\alpha). \tag{18}$$

(It is natural to rescale the soliton density by the phonon mass because $m\sim\hbar$ and the BA soliton density is in quantum phase space, whereas the classical soliton density is per unit length.) The rescaling (18) leads to the following equations for the phonon and soliton density functions:

$$u(\alpha)=\cosh\alpha+\frac{t}{2\pi}\int_{-\infty}^{\infty}dx\cos(\alpha x)x\tanh\left(\frac{\pi x}{2}\right)\times\int_{-\infty}^{\infty}d\alpha'\cos(\alpha' x)\ln\left\{1-\exp\left[-\frac{m}{T}u(\alpha')\right]\right\}+\frac{2}{\pi}\int_{-\infty}^{\infty}d\alpha'\frac{g(\alpha')}{\cosh(\alpha'-\alpha)},\tag{19}$$

$$\ln g(\alpha)=-\frac{1}{t}\cosh\alpha+\frac{1}{\pi}\int_{-\infty}^{\infty}d\alpha'\frac{\ln u(\alpha')}{\cosh(\alpha-\alpha')}-\frac{1}{\pi t}\int_{-\infty}^{\infty}d\alpha'\ln\left(\frac{\cosh(\alpha'-\alpha)+1}{\cosh(\alpha'-\alpha)-1}\right)g(\alpha').$$

It is now straightforward in principle, although quite tedious in practice, to solve these equations by iteration in the soliton density $e^{-1/t}$ and the temperature $t$. The resulting expressions for $u(\alpha)$ and $g(\alpha)$ are then substituted in the following expression for the free energy from (5) and (13):

$$F=-\frac{mM}{T}\int_{-\infty}^{\infty}d\alpha\cosh\alpha g(\alpha)+\frac{mMt}{2\pi}\int_{-\infty}^{\infty}d\alpha\cosh\alpha\ln\left\{1-\exp\left[-\frac{m}{T}u(\alpha)\right]\right\}.\tag{20}$$

Maki did the first steps in the $e^{-1/t}$ iteration in his paper. We have carried out a far more extensive analysis including the phonon-phonon interactions and find

$$F=F_0+F_1+F_2+O(e^{-3/t}),$$

where

$$F_0=F_{\text{non}}+\frac{mM}{8}\left(-2t^2-t^3-\frac{3}{2}t^4-\frac{53}{16}t^5-\frac{297}{32}t^6+O(t^7)\right),$$

$$F_1=-2mM\left(\frac{2t}{\pi}\right)^{1/2}e^{-1/t}\left(1-\frac{7}{8}t-\frac{59}{128}t^2-\frac{897}{1024}t^3-\frac{75\,005}{32\,778}t^4+O(t^5)\right),\tag{21}$$

$$F_2=\frac{8mM}{\pi}e^{-2/t}\left[\ln\frac{4\gamma}{t}-\frac{5}{4}t\left(\ln\frac{4\gamma}{t}+1\right)-\frac{1}{32}t^2\left(13\ln\frac{4\gamma}{t}+2\right)+O(t^3)\right].$$

$F_{\text{non}}$ is the free energy of the noninteracting phonons given by Eq. (10), and $C=\ln\gamma$ is the Euler constant.

The details of our work will be presented elsewhere. In comparing our results with the recent classical transfer-matrix work of Sasaki, we find that all the nonlinear terms calculated in $F_0$ and $F_1$ agree exactly. In the bracket in the last term of $F_2$ Sasaki gets $13\ln(4\gamma/t)+4$. This discrepancy may be a computational error (both methods require very long calculations for this term), or may be a signal that our possible neglect of heavy breathers enters at this order. In any case, it is clear that our soliton-anharmonic-phonon equations (19) represent very accurately the thermodynamics of the sine-Gordon system in the classical limit.

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