Threshold Behavior and Levinson's Theorem for Two-Dimensional Scattering Systems: A Surprise

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The low-energy threshold behavior and Levinson's theorem are derived for general, not necessarily rotationally symmetric, two-dimensional scattering systems, with zero-energy resonances and zero-energy bound states explicitly taken into account. Surprisingly, *s*-wave–type zero-energy resonances do not contribute at all to Levinson's theorem, while *p*-wave–type zero-energy resonances each contribute a term $-\pi$, exactly like (zero-energy) bound states. Some consequences of this for, e.g., the Witten index in certain supersymmetric quantum-mechanical models are briefly discussed.

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Two-dimensional scattering systems have become an important object of current interest. Recently, the well-known low-energy concepts of scattering length and effective range have been generalized to $n \geqslant 2$ dimensions.[1] This has given new insights into low-energy phenomena, e.g., in spin-polarized quantum systems. In many of these low-energy phenomena the occurrence of zero-energy resonances or zero-energy bound states plays a significant role. For example, recently it has been suggested[2] that such a state might be responsible for the fast surface recombination rate for deuterium atoms below 1 K. We also recall the three-body Efimov effect,[3,4] where the appearance of such states in the underlying two-body subsystems causes an infinite number of three-body bound states. Another manifestation of this role can be found in the value of the Witten index[5] for supersymmetric quantum-mechanical models. In particular, in one dimension it has been shown that a zero-energy state contributes a term $\pm\frac{1}{2}$ to this index, giving it a fractional value.[6] The arguments used in that work are based upon Levinson's theorem for one-dimensional scattering in the presence of zero-energy states.[7,8]

Reference 6 also confirms again the usefulness and importance of Levinson's theorem. In this respect, it has been shown recently that this theorem, extended to include repulsive Coulomb potentials, provides information on the nodal structure of the zero-energy wave function of the problem.[9]

It is not yet known how this fundamental Levinson's theorem extends in the presence of these zero-energy states in two dimensions. (For three dimensions, cf. Newton[10] and Osborn and Bollé.[11]) One of the reasons for this gap is that the problem is technically rather difficult because of the logarithmic singularity of the two-dimensional free Green's function at zero energy.

In this Letter we show how to overcome this difficulty. We start from the T operator defined by

$$T(k)=[\lambda_0 uR_0(k)v+1]^{-1}, \qquad \mathrm{Im}k \geqslant 0, \quad k\neq 0, \quad \lambda_0 \text{ real}, \tag{1}$$

where we have factorized the potential V as $V(\mathbf{x}) = u(\mathbf{x})v(\mathbf{x})$ with $v(\mathbf{x}) = |V(\mathbf{x})|^{1/2}$, $u(\mathbf{x}) = v(\mathbf{x}) \times \mathrm{sgn}V(\mathbf{x})$, and where λ_0 is the coupling constant. The integral kernel of the free Green's function, $R_0(k)$, in two dimensions is given by

$$R_0(k,\mathbf{x},\mathbf{y})=\tfrac{1}{4}iH_0^{(1)}(k|\mathbf{x}-\mathbf{y}|), \quad \mathbf{x}\neq\mathbf{y}, \tag{2}$$

with $H_0^{(1)}$ the Hankel function of the first kind. (We use natural units and put the mass $m=\frac{1}{2}$.) This expression has a logarithmic singularity at $k=0$. The trick is then to define an operator $M(k)$ as follows[12]:

$$uR_0(k)v=(2\pi)^{-1}[-\ln k+i\pi/2+\ln 2+\Psi(1)]|u\rangle\langle v|+M(k), \quad \mathrm{Im}k\geqslant 0, \quad k\neq 0, \tag{3}$$

with Ψ the digamma function. This operator $M(k)$ has a convergent two-variable expansion with respect to $1/\ln k$ and $k^2\ln k$ around $k=0$ if, roughly speaking, the potential is exponentially decreasing at infinity.[13] In particular the kernel of $M(0)=M_{00}$, the first term in this expansion, reads

$$M_{00}(\mathbf{x},\mathbf{y}) = -(2\pi)^{-1}u(\mathbf{x})\ln|\mathbf{x}-\mathbf{y}|v(\mathbf{y}), \quad \mathbf{x}\neq\mathbf{y}. \tag{4}$$

A low-energy analysis of $T(k)$ strongly depends, of course, on the possible occurrence of zero-energy states of the Hamiltonian of the system. To describe these states we introduce the functions ϕ as solutions of[12]

$$\lambda_0 Q M_{00} Q\phi = -\phi, \quad \phi\in L^2(R^2), \quad Q=1-P, \quad P=(v,u)^{-1}|u\rangle\langle v|. \tag{5}$$

Indeed, then it can be shown that the zero-energy Schrödinger equation is satisfied for the functions

$$\psi(\mathbf{x}) = -(v,u)^{-1}\lambda_0(v,M_{00}\phi) - (2\pi)^{-1}\lambda_0\int d^2y\,\ln|\mathbf{x}-\mathbf{y}|v(\mathbf{y})\phi(\mathbf{y}), \tag{6}$$

with the additional property $u(\mathbf{x})\psi(\mathbf{x}) = -\phi(\mathbf{x})$. Furthermore, looking essentially at the large-$|\mathbf{x}|$ behavior of the wave functions $\psi(\mathbf{x})$, one has

$$\psi(\mathbf{x}) + (v,u)^{-1}\lambda_0(v,M_{00}\phi) - (2\pi)^{-1}\lambda_0|\mathbf{x}|^{-2}\int d^2y\,(\mathbf{x}\cdot\mathbf{y})v(\mathbf{y})\phi(\mathbf{y}) \in L^2(R^2). \tag{7}$$

This property is important to distinguish between bound-state wave functions $[\psi(\mathbf{x})\in L^2(R^2)]$ and resonance wave functions $[\psi(\mathbf{x})\notin L^2(R^2)]$.

Introducing the notation

$$c_1^{(j)} = (v,u)^{-1}(v,M_{00}\phi_j), \quad \mathbf{c}_2^{(j)} = (2\pi)^{-1}\int d^2x\,\mathbf{x}v(\mathbf{x})\phi_j(\mathbf{x}), \tag{8}$$

we then find the following possibilities.

Case I.—Equation (5) has no solutions or, equivalently, there exist no zero-energy states ψ.

Case II.—Equation (5) has $N\leqslant 3$ solutions ϕ_j, $0\leqslant j\leqslant 2$, satisfying $\phi_j\in L^2(R^2)$, $\psi_j\notin L^2(R^2)$, so that they are all zero-energy resonances, and (a) $N=1$, $c_1^{(0)}\neq 0$, $\mathbf{c}_2^{(0)}=0$, or $\mathbf{c}_2^{(0)}\neq 0$; or (b) $N\leqslant 2$, $c_1^{(j)}=0$, $\mathbf{c}_2^{(j)}\neq 0$, $1\leqslant j\leqslant 2$, and the $\mathbf{c}_2^{(j)}$ are linearly independent; or (c) $2\leqslant N\leqslant 3$, $c_1^{(0)}\neq 0$, $\mathbf{c}_2^{(0)}=0$, or $\mathbf{c}_2^{(0)}\neq 0$, $c_1^{(j)}=0$, $\mathbf{c}_2^{(j)}\neq 0$, $1\leqslant j\leqslant 2$, and the $\mathbf{c}_2^{(j)}$ are linearly independent.

Case III.—Equation (5) has N solutions ϕ_j, N finite, $3\leqslant j\leqslant N+3$, satisfying $\phi_j\in L^2(R^2)$, $\psi_j\in L^2(R^2)$, so that $c_1^{(j)}=\mathbf{c}_2^{(j)}=0$ and they are all zero-energy bound states.

Case IV.—Admixtures of cases II and III.

The way in which property (7) is satisfied in case II, giving rise to the possibilities II(a) to II(c), is important. Indeed, if the potential V is rotationally symmetric, one can explicitly show that case II(a) corresponds to an s-wave zero-energy resonance and case II(b) to p-wave zero-energy resonances. Case III corresponds to zero-energy bound states appearing in the d and higher partial waves. We remark that for reasons of brevity none of the cases IV will be discussed here (cf. Ref. 13).

Then we can prove that in all cases $T(k)$ has a convergent two-variable Laurent expansion with respect to $1/\ln k$ and $k^2\ln k$ around $k=0$.[13] We only explicitly specify the singular behavior of $T(k)$: case I,

$$T(k) = (1+\lambda_0 Q M_{00} Q)^{-1}Q + O(1/\ln k); \tag{9}$$

case II(a),

$$T(k) = \ln k\,(2\pi\lambda_0|c_1^{(0)}|^2)^{-1}|\phi_0\rangle\langle\tilde{\phi}_0|; \tag{10}$$

case II(b),

$$T(k) = -(k^2\ln k)^{-1}(\pi\lambda_0)^{-1}\sum_{j,l=1}^{N}(\mathbf{c}_2^{(j)}\cdot\mathbf{c}_2^{(l)})^{-1}|\phi_j\rangle\langle\tilde{\phi}_l| + O(1); \tag{11}$$

case III,

$$T(k) = k^{-2}\lambda_0^{-1}\sum_{j,l=3}^{N+3}(\tilde{\phi},M_{11}\phi)_{jl}^{-1}|\phi_j\rangle\langle\tilde{\phi}_l| - \ln k\sum_{j,l=3}^{N+3}(\tilde{\phi},M_{11}\phi)_{jl}^{-1}|\phi_j\rangle\langle\tilde{\phi}_l|M_{01}T_0Q + O(1); \tag{12}$$

with

$$M_{01}(\mathbf{x},\mathbf{y}) = (8\pi)^{-1}u(\mathbf{x})|\mathbf{x}-\mathbf{y}|^2 v(\mathbf{y}),$$

$$M_{11}(\mathbf{x},\mathbf{y}) = -(16\pi)^{-1}[(i\pi+\ln 4+\Psi(2)u(\mathbf{x})|\mathbf{x}-\mathbf{y}|^2 v(\mathbf{y}) - 2u(\mathbf{x})(\ln|\mathbf{x}-\mathbf{y}|)|\mathbf{x}-\mathbf{y}|^2 v(\mathbf{y})],$$

$$T_0 = \lim_{\epsilon\to 0}(1+\lambda_0 Q M_{00} Q+\epsilon)^{-1}(1-P_0),$$

where P_0 denotes the following projector:

$$P_0 = \sum_{j=1}^{N}(\tilde{\phi}_j,\phi_j)^{-1}|\phi_j\rangle\langle\tilde{\phi}_j|, \quad \tilde{\phi}_j = (\operatorname{sgn}V)\phi_j, \quad (\tilde{\phi}_j,\phi_l) = \delta_{jl}(\tilde{\phi}_j,\phi_j).$$

We remark that the singular behavior of $T(k)$ in case II(c) is given by the sum of the singular terms in cases II(a) and II(b). It is also interesting to note that the s-wave type zero-energy resonance causes a (weak) logarithmic singularity in $T(k)$. These results are new.

Next we use these results to discuss the low-energy properties of $\mathrm{Tr}[R(k)-R_0(k)]$. This quantity is a relevant one to look at for deriving Levinson's theorem, e.g., since it is connected with the on-shell S matrix $S(k)$ in the following way[13,14]:

$$2\,\mathrm{Im}\,\mathrm{Tr}[R(k)-R_0(k)] = -i(2k)^{-1}\mathrm{Tr}[(d/dk)\ln S(k)]. \tag{13}$$

Employing

$$\mathrm{Tr}[R(k)-R_0(k)] = -\lambda_0 \mathrm{Tr}[R_0(k)vT(k)uR_0(k)] \tag{14a}$$

$$= -\lambda_0(2k)^{-1}\mathrm{Tr}[uR_0'(k)v] + \lambda_0^2(2k)^{-1}\mathrm{Tr}[uR_0'(k)vT(k)uR_0(k)v], \tag{14b}$$

we can show that the function $\mathrm{Tr}[R(k)-R_0(k)]$ has again a two-variable Laurent expansion in $1/\ln k$ and $k^2\ln k$ around $k=0$. Here we only need the coefficient of the term $(1/\ln k)^{-1}(k^2\ln k)^{-1} = k^{-2}$, viz.,

$$\mathrm{Tr}[R(k)-R_0(k)] = -Dk^{-2} + O((k^2\ln k)^{-1}), \tag{15}$$

where

$$D^{\mathrm{I}}=0,\quad D^{\mathrm{II(a)}}=0,\quad D^{\mathrm{II(b)}}=N,\quad D^{\mathrm{II(c)}}=(N-1),\quad D^{\mathrm{III}}=N. \tag{16}$$

Equations (13)–(16) provide all the necessary information to derive Levinson's theorem by the standard method of contour integration in the complex energy plane. The analytic function to start from is

$$f(k^2) = \mathrm{Tr}[R(k)-R_0(k)] + Dk^{-2} - \lambda_0 k^{-2}\int d^2x\, V(\mathbf{x}).$$

Here the last term on the right-hand side stems from the high-energy Born behavior of $\mathrm{Tr}[R(k)-R_0(k)]$ [cf. Eq. (14b)]. The contour integration then yields

$$\int_0^\infty dk^2\{-i(2k)^{-1}\mathrm{Tr}[(d/dk)\ln S(k)]\} = -2\pi N_b - 2\pi D - (\lambda_0/2)\int d^2x\, V(\mathbf{x}), \tag{17}$$

where N_b is the number of negative-energy bound states, where D is given by Eq. (16), and where we have assumed that there are no eigenvalues of the Hamiltonian embedded in the continuum.

If V is rotationally symmetric, Eq. (17) for a fixed partial wave l leads to

$$\delta_l(\infty) - \delta_l(0) = -\pi N_{b,l} - \pi D_l, \tag{18}$$

where δ_l is the phase shift, $D_0=0$, and $D_l=0$ or 1, $l \geqslant 1$.[15] On the right-hand side of (18), a Born-term subtraction is absent because of the better high-energy behavior of the corresponding fixed partial wave quantities.

These formulas extend Levinson's theorem in the presence of zero-energy states to two dimensions. [In case I, Eq. (17) is consistent with Osborn *et al.*[16] and Cheney.[17]] These results are surprising. Indeed, we know that in three dimensions there possibly exists only a zero-energy resonance of the s-wave type and it contributes a term $-\pi/2$ to the corresponding Levinson's theorem.[10,11] A similar result holds for the zero-energy resonance in one dimension.[7,8] Here we find that in two dimensions a possible s-wave–type zero-energy resonance does *not* contribute at all to this theorem, while there possibly exist also p-wave–type zero-energy resonances that contribute each a term $-\pi$, exactly like (zero energy) bound states.

Of course, this must have some consequences in certain two-dimensional phenomena. As an example of current interest, we briefly discuss supersymmetric quantum-mechanical models, where one studies Hamiltonians of the form $H=Q^2$ with Q the supersymmetric charge

$$Q = \begin{pmatrix} 0 & L \\ L^\dagger & 0 \end{pmatrix}.$$

One of the challenging problems in this field is to detect mechanisms by which this supersymmetry can be spontaneously broken. For this purpose Witten introduced[5] an order parameter Δ, counting the difference of the number of bosonic and fermionic zero-energy modes of H. If this Witten index $\Delta \neq 0$, which is an indication of the existence of zero-energy states, then supersymmetry is not broken.

For one-dimensional models it has been found (cf., e.g., the references in Ref. 6) that Δ, appropriately regularized, may be fractional when the continuous spectrum in the model extends to zero energy. Recent calculations, e.g., Ref. 6, have shown through use of Jost-function techniques that Δ is given in terms of the relative phase shift of the associated operators $LL^\dagger$ and $L^\dagger L$. By employing the corresponding Levinson's theorem these authors find that these fractional values, $\pm\frac{1}{2}$, exactly arise from the contribution of a zero-energy resonance to this theorem. So this indicates that there is a zero-energy state in these models and hence supersymmetry is unbroken. For three-dimensional spherically symmetric models it is claimed that the same conclusion holds.[6] We remark that an explicit construction of such spherically symmetric models in n dimensions is given in the work of Imbo and Sukhatme.[18]

The results on the two-dimensional Levinson's theorem we have obtained here at first sight seem to imply, if one makes a similar reasoning, that the Witten index is never fractional in these two-dimensional supersymmetric quantum-mechanical models with rotational symmetry, in contrast with one and three dimensions. However, a closer investigation, taking into account the long-range relative interaction potential between $LL^\dagger$ and $L^\dagger L$ (of the type c/r^2, $c\neq 0$) shows that fractionalization can occur if the long-distance behavior in L is trivial.[19]

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[14]See also M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964), Sect. 8.5.

[15]Using Jost-function techniques and the results of Ref. 1, one can generalize Eq. (18) to n dimensions as follows (Ref. 13):

$$\delta_l^{(n)}(\infty)-\delta_l^{(n)}(0)=-\pi N_{b,l}^{(n)}-\pi D_l^{(n)},\quad n\geq 2,\ l\geq 0,$$

where $D_0^{(2)}=0$, $D_l^{(2)}=0$ or 1, $l\geq 1$; $D_0^{(3)}=0$ or $\frac{1}{2}$, $D_l^{(3)}=0$ or 1, $l\geq 1$; $D_l^{(n)}=0$ or 1 for $n\geq 4$, $l\geq 0$.

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