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Must Nonspherical Collapse Produce Black Holes? A Gravitational Confinement Theorem

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It is shown that a trapped surface can be extended, in a locally area-preserving fashion, to a three-cylinder that is everywhere spacelike, provided it encounters no singularities in its development. Thus the interior of a regular, area-preserving, initially trapped surface is permanently sealed off from causal influence on the exterior world.

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In the familiar scenario for the gravitational collapse of an exactly spherical star, an event horizon forms once matter has shrunk within its Schwarzschild radius. The horizon has the crucial role of preserving asymptotic predictability: It acts as a dam preventing inundation of the environment by uncontrollable emissions of radiation and other causal influences from the infinitely compressed material. An —obviously very desirable—assurance that this idealized scenario is actually generic would come from a proof of some kind of "event-horizon conjecture" (EHC).[1] Broadly stated, EHC would require that an event horizon must develop whenever a matter distribution, whose energy density satisfies appropriate positivity conditions, has collapsed to the point where a trapped surface forms. (A trapped surface is a closed two-space with the property that narrow beams of light orthogonal to it at any point decrease in area, at least initially, whether propagating inwards or outwards.) So far, EHC basically remains a pious hope, bolstered by studies of simple special cases and a failure to find counterexamples.

It has been traditional to impose a narrower interpretation on EHC by focusing on the singularities that develop in the collapse—according to a 1965 theorem of Penrose,[2] there must be at least one—and conjecturing ("cosmic censorship"[3–5]) that no such singularity can be "naked," i.e., not enclosed within a horizon, at least if it is sufficiently "strong"[3] or stable under generic perturbations of initial data and equations of state.[6]

Cosmic censorship, as presently formulated, confronts a growing body of counterevidence (or, at any rate, constraints) in the shape of naked central singularities that develop by "shell focusing" in models of anhomologous spherical collapse.[7] They may be visualized as "singular past horizons," and can be either transient or persistent in terms of external retarded time. They are singularities of infinite curvature, although massless in a sense that is well defined for spherical symmetry. These models are not counterexamples to EHC, however, since event horizons do form around the massive central singularities that generally develop afterwards.

The aspect of EHC to be addressed here circumvents the delicate issue of singularities. In brief, the question is as follows: Will a trapped surface S_0 always remain trapped? In other words, can one prescribe a future history for S_0 that will extend it, more or less rigidly, to a three-cylinder Σ that is everywhere spacelike, so that later sections of Σ are also trapped? (I shall expressly assume that Σ itself encounters no singularities in its development, though it will generally enclose at least one and there may be lesser singularities due, e.g., to "shell crossing" exterior to it.) If this can be shown, it would guarantee that whatever lurks inside S_0 is permanently sealed off from causal influence on the external world, barring some violent eruption. (Since past light cones on Σ face outward, any such eruption would presumably entail a gross violation of causality on Σ.) A confinement property of this type seems to capture much of the essence of EHC, and may be as much as one can reasonably hope to prove with presently available techniques.

In seeking to implement the notion that the region causally sealed off by S_0 should retain a "fixed size," one has to bear in mind that it would be overly stringent to demand ridigity of the intrinsic two-geometry of S_0 in a space-time whose curvature varies arbitrarily; e.g., a large eggshell in such a space-time would crack. We shall instead require that the evolution of S_0 be locally area preserving, which means that not only the total area of S_0, but also the areas of all surface elements out of which it is composed, are required to remain constant.

To set up an appropriate formal machinery, suppose that a closed, convex two-space S_0 is Lie-transported along integral curves of a space-time vector field ξ^α, parametrized by t with $t=0$ on S_0. This generates a three-cylinder Σ, with parametric equations $x^\alpha = x^\alpha(\theta^a, t)$ in terms of Lie-transported two-dimensional coordinates θ^a $(a=2,3)$ (Fig. 1). The outgoing and incoming future lightlike vectors $l^\alpha_{(A)}$ $(A=0,1)$, which generate the pair of null hypersurfaces orthogonal to any two-section $S(t)$, may be partially normalized by $l^\alpha_{(A)} l_{(B)\alpha} = \eta_{AB}$, where $\eta_{AB} =$ antidiag$(-1,-1)$ will be used to lower and raise uppercase Latin indices. One can then decompose

$$\xi^\alpha = n^\alpha + \xi^a e^\alpha_{(a)}, \quad n^\alpha = \xi^A l^\alpha_{(A)},$$

where $e^\alpha_{(a)} = \partial x^\alpha(\theta^a, t)/\partial\theta^a$ are coordinate basis vectors of $S(t)$ and n^α is a vector in Σ orthogonal to $S(t)$. If g_{ab} is the intrinsic two-metric of $S(t)$, it is easy to verify that

$$\tfrac{1}{2}\partial g_{ab}/\partial t = \xi_{(a;b)} - \xi^A K_{Aab},$$

where $K_{Aab} = -l_{(A)\alpha|\beta} e^\alpha_{(a)} e^\beta_{(b)}$ are extrinsic curvatures of $S(t)$; their traces, K_0 and K_1, measure the convergence rates of outgoing and incoming light beams. It follows that the variation along the normal direction n^α of the element of two-area dS is given by

$$\mathscr{L}_{n^\alpha} dS = -\xi^A K_A \, dS. \tag{1}$$

According to (1), the evolution of S_0 into a three-cylinder Σ is locally area preserving if the two com-

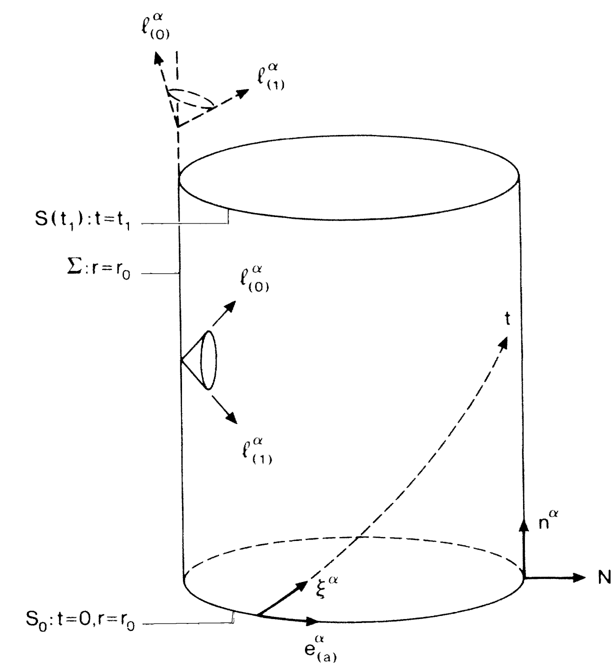

FIG. 1. The two-surface S_0 and its extension to a three-cylinder Σ.

ponents of ξ^α transverse to $S(t)$ are required to satisfy

$$\xi^A K_A = 0. \tag{2}$$

The remaining arbitrariness of these components resides in the overall scaling of ξ^α, i.e., in the definition of t. This freedom may be partially constrained by requiring that *ingoing* light beams orthogonal to $S(t)$ should always converge, and that their convergence rate should be uniformly bounded away from zero. This condition simultaneously serves to prevent a locally area-preserving Σ from becoming tangent to ingoing light rays and thus to preserve a regular monotonic relationship between t and exterior advanced time. The mathematical expression of this condition is that the function

$$C(\theta^a, t) = (-l^\alpha_{(1)|\alpha})/(-l^\alpha_{(1)}\xi_\alpha) = K_1/\xi^0, \tag{3}$$

which is independent of the normalization of the ingoing null geodesics, should be bounded above and below by positive functions independent of t. A locally area-preserving evolution that satisfies this constraint will be called "semirigid."

From (2) and (3),

$$K_A = -C\epsilon_{AB}\xi^B, \tag{4}$$

where ϵ_{AB} is the permutation symbol, with $\epsilon_{01} = \epsilon^{10} = +1$. The vector

$$N^\alpha = -\epsilon(N)\epsilon^{AB}\xi_A l^\alpha_{(B)}, \quad \epsilon(N) = \mathrm{sgn}(N^\alpha N_\alpha)$$

is an outward normal to Σ.

We are now in a position to state the *theorem.* Let the three-cylinder Σ be a semirigid future extension of the trapped two-surface S_0. Then Σ is initially spacelike, and must remain spacelike, at least as long as it remains nonsingular. It is assumed that any matter crossing Σ satisfies a weak energy condition ($T_{\alpha\beta}u^\alpha u^\beta \geqslant 0$ for all timelike u^α).

The first part of the theorem is geometrically obvious. To prove it formally, one notes that, initially, $K_0 > 0$ (since S_0 is trapped) and $K_1 > 0$ always. It then follows from (2) that, initially, ξ^1 is positive and $N^\alpha N_\alpha = -\xi^A \xi_A$ is negative.

We next show that $N^\alpha N_\alpha$ can never become positive by passing through zero. To this end, we invoke one of the Arnowitt-Deser-Misner constraint equations for Σ:

$$\epsilon^{AB}\dot{K}_A \xi_B = 8\pi\epsilon(N) T^{\alpha\beta} n_\alpha N_\beta + \epsilon_{AB}({}^{(2)}\nabla^2 \xi^A)\xi^B + (\xi^A \xi_A \sigma^a)_{;a} + \epsilon^{AB}\xi^a(\partial_a K_A)\xi_B + \sigma K_A \epsilon^A + \epsilon^{AB} K_{Aab} K_D^{ab} \xi^D \xi_B, \quad (5)$$

where $\sigma = l_{(0)\alpha|\beta} l^\alpha_{(1)} n^\beta$, $\sigma_a = l_{(0)\alpha|\beta} l^\alpha_{(1)} e^\beta_{(a)}$, and a dot denotes the "time" derivative $\partial/\partial t$. Suppose that there were a transition point p_1, on a section $S(t_1)$, at which N^α first became null. At p_1 we should then have

$$n^\alpha = -N^\alpha = l^\alpha_{(0)}, \quad \xi^A \xi_A = \partial_a(\xi^A \xi_A) = 0,$$

$$\xi^1 = \partial_a \xi^1 = 0, \quad {}^{(2)}\nabla^2 \xi^1 \leqslant 0.$$

By insertion of (4) it is now easy to show that each individual term on the right-hand side of (5) is zero or positive at p_1. Thus (5) reduces to

$$\tfrac{1}{2} C \partial(\xi^A \xi_A)/\partial t|_{p_1} \geqslant 0,$$

which contradicts our hypothesis that $\xi^A \xi_A$ has reached zero from positive values and is about to pass to negative values. The essence of this result can be understood geometrically on the basis of Raychaudhuri's equation.[8] Because of the focusing effects of shear and matter, an outgoing pencil of light orthogonal to $S(t_1)$, which must be momentarily stationary at the hypothetical transition point p_1, cannot become expanding to the future of p_1. Therefore, an area-preserving surface element of Σ cannot slice the light cones to the future of p_1 and thus its history cannot pass from spacelike to timelike at p.

A fuller account of this work is scheduled for publication elsewhere.[5]

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[8]Kindly drawn to my attention by R. Wald. For Raychaudhuri's equation, see, e.g., Ref. 4, p. 218.