Universal Term in the Free Energy at a Critical Point and the Conformal Anomaly

Ian Affleck
Department of Physics, Princeton University, Princeton, New Jersey 08544
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We show that the leading finite-size correction to $\ln Z$ for a two-dimensional system at a conformally invariant critical point on a strip of length L, width β ($\beta << L$), is $(\pi/6)c(L/\beta)$, where c is the conformal anomaly. Equivalently, the leading low-temperature correction to the free energy of a one-dimensional quantum system is $-(\pi/6)cL(kT)^2/\hbar v$, where v is the effective "velocity of light." The latter formula is used to check recently derived critical theories of spin-s quantum chains against Bethe-*Ansatz* solutions.

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Conformal invariance powerfully constrains the critical behavior of two-dimensional classical (and one-dimensional quantum) systems.[1,2] Critical theories are parametrized by the conformal anomaly c, which is the central charge in the Virasoro algebra obeyed by the energy-momentum tensor:

$$-i[T(x_-),T(x'_-)]=\delta(x_- - x'_-)T' - 2\delta'(x_- - x'_-)T + (c/24\pi)\delta'''(x_- - x'_-) \qquad [T=(T_{00}-T_{01})/2]. \qquad (1)$$

For $c<1$ a discrete set of values are allowed by unitarity[2] (reflection positivity): $c=1-6/m(m+1)$, $m=3,4,5,\ldots$. These are realized by the Ising ($c=\frac{1}{2}$), tricritical Ising ($c=\frac{7}{10}$), three-state Potts ($c=\frac{4}{5}$), tricritical three-state Potts ($c=\frac{6}{7}$), and other models. A complete classification has not been given for $c\geqslant 1$ except when a continuous symmetry G is assumed.[3,4] For $G=\mathrm{U}(1)$ we get the Gaussian model ($c=1$) which describes a wide variety of critical phenomena ($q=4$ Potts model, X-Y model, Coulomb gas, $s=\frac{1}{2}$ antiferromagnet, ...) For $G=\mathrm{SU}(n)$ the possible values of c are $(n^2-1)k/(n+k)$, $k=1,2,3,\ldots$. These describe Wess-Zumino σ models[5] and antiferromagnetic chains (and perhaps also two-dimensional statistical models). In all the above cases, all scaling dimensions are known exactly.

Conformal invariance can also be used to study finite-size effects in two-dimensional statistical systems[6] or finite-temperature effects in one-dimensional quantum systems. These are related because a $(1+1)$-dimensional quantum field theory at temperature T is given by a Euclidean-space functional integral on a strip of width $\beta=1/T$ (in the imaginary time direction). Correlation functions behave as

$$\langle\phi(x,0)\phi(y,0)\rangle \sim e^{-|x-x|/l_\phi} \qquad (2)$$

with $l_\phi=2\pi d_\phi/\beta$, where d_ϕ is the scaling dimension of of ϕ and periodic boundary conditions are imposed on strip of width β.[6] This has been generalized to other boundary conditions and the interfacial tension (difference in free energy per unit length for periodic and antiperiodic boundary conditions) has been related to the scaling dimension of a disorder operator, in some cases.

In this work, we will derive a simple, general formula for the leading finite-width correction to $\ln Z$:

$$(\ln Z)/L=\text{const}\times\beta+\pi c/6\beta+O(1/\beta^2). \qquad (3)$$

Here β is the width of a strip with periodic boundary conditions and c is the conformal anomaly. (The length L is taken to infinity.) Equivalently, for a one-dimensional quantum system we obtain the same formula but now $1/\beta=T$ and the correction is scaled by the effective "velocity of light" v (which occurs in the low-energy excitation spectrum):

$$F/L=\epsilon_0-\pi c T^2/6v+O(T^3). \qquad (4)$$

(We set $\hbar$ and Boltzmann's constant equal to 1.) Note that for the statistical problems (assumed to be defined on a square lattice with rationally invariant couplings) the speed of light is $v=1$.

The proof of this result rests on the definition of c as the response of a theory to curving of the two-dimensional space. If Z is the partition function on a space with metric $q_{\mu\nu}$ then[7]

$$-g^{\mu\nu}\frac{\delta\ln Z}{\delta g^{\mu\nu}}=g^{\mu\nu}\langle T_{\mu\nu}\rangle=\frac{c}{48\pi}[R(x)+\mu^2]. \qquad (5)$$

Here $T_{\mu\nu}$ is the energy-momentum tensor and the first equation follows from the canonical definition of $T_{\mu\nu}$. $R(x)$ is the curvature scalar and the second equation follows because R is the only invariant function of the metric with the right dimension (L^{-2}). μ^2 is a constant (of dimension L^{-2}) and c is an arbitrary constant but it can be shown[8] to be the same one which appears in the Virasoro algebra by variation of $\ln Z$ a second time with respect to the metric:

$$\langle T(x)T(x')\rangle = c/2(x-x')^4+\text{less singular terms}. \qquad (6)$$

But this leading singularity in the operator product expansion can also be determined from the Virasoro algebra and thus the constant c must be the same. Equation (5) can now be integrated to find[7]

$$-\ln Z = (c/48\pi)\int d^2x\,(\tfrac{1}{2}\,\partial_a \ln\rho\,\partial_a \ln\rho + \mu^2\rho) + \alpha. \tag{7}$$

Here we have chosen the metric in conformal gauge, $g_{\mu\nu} = \rho(x)\delta_{\mu\nu}$, and α is a dimensionless constant. We assume that the manifold has no boundaries so that there are no "surface terms" to worry about.

Let us now fix the space Γ_0 to be a strip of length L in the x direction, $0 \leqslant x \leqslant L$ ($L \to \infty$), and width β. We assume periodic boundary conditions in the y direction. Boundary conditions in the x direction are immaterial because $L \to \infty$. In what follows it is actually convenient to impose Dirichlet boundary conditions in the x direction. Equation (7) would then be corrected by (L independent) boundary terms. If we consider an arbitrary manifold Γ that can be obtained from Γ_0 by a conformal transformation $w = f(z)$ then $\ln Z_\Gamma$ is given by $\ln Z$ on Γ_0 with a metric $\rho = |\partial f/\partial Z|^2$. Thus[9]

$$\ln Z_{\Gamma_0} - \ln Z_\Gamma = (c/48\pi)\int d^2x\,[\tfrac{1}{2}(\partial_a \ln\rho)^2 + \mu^2(\rho - 1)].$$

Consider now $W = e^{-2\pi z/\beta}$. This maps Γ_0 onto Γ, the annulus of outer radius 1, inner radius $e^{-2\pi L/\beta} \to 0$, with Dirichlet boundary conditions. (Note that the boundaries of Γ_0 at $y = \pm\beta/2$ are mapped onto the same line in Γ.) We expect $\ln Z_\Gamma$ to remain finite as $L \to \infty$ (Γ simply becomes the unit disk with vanishing conditions at the origin and its boundary). So we conclude that

$$\lim_{L\to\infty} \frac{\ln Z_{\Gamma_0}}{L} = \text{const} \times \beta + \frac{c\pi}{6\beta}.$$

The first term, proportional to the area, is nonuniversal (depends on μ^2), but the second *is* universal, depending only on the conformal anomaly, c.

We can immediately check this formula by applying it to the Gaussian model with $c = 1$. The simplest way of doing this is to use the equivalence of the classical partition function on a strip of width β with the one-dimensional quantum partition function at temperature $T = 1/\beta$. Thus

$$\frac{\ln Z_G}{L} = -\frac{F}{TL} = -\frac{\epsilon_0}{T} - \int_{-\infty}^{\infty} \frac{dp}{2\pi} \ln[1 - e^{-|p|/T}]$$
$$= -\frac{\epsilon_0}{T} + \frac{\pi}{6}T. \tag{8}$$

The ground-state energy per unit length is ultraviolet-cutoff dependent but the leading T-dependent part of F is universal. [It determines the heat capacity per unit length, $C/L = (\pi/3)T$.] Another simple check is the Ising model at the critical point ($c = \frac{1}{2}$). The partition function on the strip is that of a one-dimensional system of free fermions[10] with dispersion relation

$$\epsilon(p) = 2|\sinh^{-1} \sin p/2| = |p| + O(p^3). \tag{9}$$

In the sector with an even fermion number the allowed values of p are $(n + \frac{1}{2})2\pi/L$ ($n = 0, \pm1, \pm2$) and in the sector with an odd fermion number they are $n2\pi/L$ ($n = 0, +-1, +-2\ldots$). However in the limit $L \to \infty$ sums over discrete moments are replaced by integrals so that this feature can be ignored, giving

$$\frac{\ln Z}{L} = -\frac{\epsilon_0}{T} + \int \frac{dp}{2\pi} \ln(1 + e^{-\epsilon(p)/T}). \tag{10}$$

Extracting the term linear in T gives

$$\frac{\ln Z}{L} = -\frac{\epsilon_0}{T} + \int_{-\infty}^{\infty} \frac{dp}{2\pi} \ln[1 + e^{-|p|/T|}] + O(T^3); \tag{11}$$

note that the second term is $\ln Z$ for a relativistic *Majorana* fermion (no antiparticle). Evaluating the integral gives

$$\frac{\ln Z}{L} = -\frac{\epsilon_0}{T} + \frac{\pi T}{12} + O(T^3) \tag{12}$$

in agreement with Eq. (3). The equivalent term for the three-state Potts model, tricritical Ising model, etc., can be read off. We see that the conformal anomaly c can be directly measured experimentally!

As a nontrivial application of this result let us consider antiferromagnetic quantum spin chains,

$$H = \sum_{n=1}^{N} P(\mathbf{S}_n \cdot \mathbf{S}_{n+1}), \quad \mathbf{S}_n^2 = s(s+1), \tag{13}$$

where P is some polynomial of degree $\leqslant 2s$. It was argued elsewhere[11] that for choices of P such that H is antiferromagnetic and has gapless excitations, it is described at low energies by the SU(2), $k = 2s$ Wess-Zumino σ model [equivalently by the SU(2) Kac-Moody algebra with central charge $k = 2s$]. For this model[13] $c = 3s/(1+s)$. The spin chain has a relativistic low-energy behavior with some effective (nonuniversal) "speed of light" or "Fermi velocity" v. This enters the universal term in the free energy in a manner determined by dimensional analysis. Thus, the low-temperature heat capacity should behave as

$$\frac{C}{L} = \frac{\pi s T}{(1+s)v} + O(T^3). \tag{14}$$

Note that the $s = \frac{1}{2}$ chain is equivalent, at low energies, to a free boson as argued long ago.[12] However, the higher-spin chains have a specific heat which is, in general, a fractional multiple of that for a free boson, demonstrating that the higher-s models contain interacting bosons. For each value of s there is one choice of polynomial P which renders H integrable[13]:

$$P(\mathbf{S}\cdot\mathbf{S}') = \sum_{l=1}^{2s} 2sa_l P_l, \tag{15}$$

where P_l is a projector onto total spin l and $a_l = \sum_{k=1}^{l} 1/k$. Comparison of the Bethe-*Ansatz* solution with the proposed critical theory is difficult because one approach gives only the spectrum and the other only the Green's functions. However, the velocity of light and low-temperature heat capacity are known from the Bethe *Ansatz*, allowing a check on the critical theory.[14] These are[15]

$$V = \pi/2 \quad [\text{for all } s],$$

$$\frac{C}{L} = \tfrac{2}{3}T \quad (s=\tfrac{1}{2}),$$

$$= T \quad (s=1),$$

$$= \tfrac{2}{3}T - \frac{2T}{\pi^2}\sum_{n=1}^{2s-1}\int_0^{a_n}\left(\frac{1}{x}\ln(1-x) + \frac{1}{1-x}\ln x\right) \quad (s \geqslant \tfrac{3}{2});$$

$$a_n = \sin^2[\pi/2(s+1)]/\sin^2[\pi(n+1)/2(s+1)].$$

We see that the exact values for $s=\frac{1}{2}, 1$ agree with our prediction. The indefinite integral cannot be evaluated exactly and so we have calculated it numerically for $s=\frac{3}{2}$ (five significant digits), 2, and $\frac{5}{2}$ (three significant digits), finding agreement with the prediction. (Thus the sum of definite integrals apparently can be done exactly.)

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