# Test of Scaling Exponents for Percolation-Cluster Perimeters

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The formula for the fractal dimensionality of the perimeter (hull) of a percolation cluster, $d_f'=1+1/\nu$, proposed recently by Sapoval, Rosso, and Gouyet, is shown to imply for the perimeter scaling exponents $\tau'=1+2\nu/(1+\nu)$, $\sigma'=1/(1+\nu)$, and $\gamma'=2$. Monte Carlo simulations of very large perimeter-generating walks yield $\tau'=2.143\pm0.002$ and $d_f'=1.751\pm0.002$, consistent with these predictions (on the assumption that $\nu=\frac{4}{3}$). The walks are also used to determine $p_c=0.592\,75\pm0.000\,03$ for site percolation on a square lattice.

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The perimeter of a percolation cluster is the continuous path of occupied sites at a boundary, which can be either external or internal to the cluster. Sapoval, Rosso, and Gouyet[1] have recently proposed that the fractal dimensionality of the perimeter, or hull, is related simply to the correlation-length exponent by

$$d_f'=1+1/\nu, \tag{1}$$

which, with the accepted value[2-4] $\nu=\frac{4}{3}$, yields $d_f'=\frac{7}{4}$. This prediction is supported by the direct measurements by Voss[5] of percolation perimeters, which yield $d_f'=1.76\pm0.01$, and by measurements by Kremer and Lyklema[6] of an equivalent nontrapping random walk, which give $1.764\pm0.01$. Sapoval, Rosso, and Gouyet[1] consider a diffusion front (which they argue is equivalent to a percolation perimeter), and find $1.76\pm0.02$. Theoretical arguments for (1) have recently been given by Bunde and Gouyet.[7] Perimeter fractal properties have also been related to the fractal structure of the growth sites in kinetically growing percolation clusters very recently.[8]

Here I show that (1) has interesting implications for the perimeter scaling exponents, and report on Monte Carlo measurements of some of their values. Near $p=p_c$, one expects that the perimeter distribution function $n'_{s'}$, where $s'$ is the number of occupied sites in the perimeter, obeys the scaling behavior

$$n'_{s'}\sim(s')^{-\tau'}f'(|p-p_c|(s')^{\sigma'}) \tag{2}$$

in analogy to the usual cluster scaling,[9] where $\tau'$ and $\sigma'$ are the scaling exponents. (All perimeter properties are indicated by a prime.) Because the typical perimeter length $s'\sim|p-p_c|^{-1/\sigma'}$ scales as $\xi^{d_f'}\sim(1|p-p_c|^{-\nu})^{d_f'}$, it follows that $1/\sigma'=\nu d_f'$, and therefore, by (1), that

$$\sigma'=1/(1+\nu). \tag{3}$$

Also, by virtue of the relationship[9-11] between $d_f$ and $\tau$ (generalized to perimeters), $d/d_f'=\tau'-1$, where $d=2$ is the dimensionality, it follows that

$$\tau'-1=2\nu(1+\nu). \tag{4}$$

From $\sigma'$ and $\tau'$ all the scaling exponents can be found. For example, the exponent $\gamma'$ for the mean perimeter length,

$$\chi\equiv\sum(s')^2n'_{s'}\sim|p-p_c|^{-\gamma'}, \tag{5}$$

turns out simply to be

$$\gamma'=(3-\tau')/\sigma'=2 \tag{6}$$

independent of $\nu$.

The above expression for $\gamma'$ has been found by Weinrib and Trugman[12] by considering the relation between the perimeter properties, which satisfy (2), and the cluster properties, which satisfy $n_s\sim s^{-\tau}f(|p-p_c|s^\sigma)$. Observing that $f'$ and $f$ are essentially the same functions, they deduce

$$\frac{\tau-1}{\tau'-1}=\frac{\sigma}{\sigma'}=\frac{d_f'}{d_f}=x, \tag{7}$$

where $x=\frac{12}{13}$ is the exponent of $s'\sim s^x$. Both (3) and (4) can be derived from (7) and the usual scaling relations for the cluster properties.

If we assume $\nu=\frac{4}{3}$, the exponents take on the numerical values listed in Table I. For comparison, the

TABLE I. Critical exponents for percolation clusters and their perimeters (on the assumption that $\frac{4}{3}$).

| | Perimeters | |
|---|---|---|
| Clusters | Predicted | Measured |
| $\sigma=\frac{36}{91}$ | $\sigma'=\frac{3}{7}$ | ... |
| $\tau=\frac{187}{91}$ | $\tau'=\frac{15}{7}$ | $2.143\pm0.002$ [a] |
| $d_f=\frac{91}{48}$ | $d_f'=\frac{7}{4}$ | $1.751\pm0.002$ [a] |
| | | $1.76\pm0.01$ [b] |
| $\gamma=\frac{43}{18}$ | $\gamma'=2$ [c] | $2.0\pm0.1$ [d] |

[a]This work.
[b]Reference 5.
[c]Independent of $\nu$.
[d]Reference 13.

exponents for the cluster properties (assuming $\sigma = \frac{36}{91}$ [2–4,14]) are also listed. The perimeters evidently have simpler exponents than the clusters themselves.

To test the above predictions for the exponents' values, I have carried out simulations of very large percolation perimeters on a square lattice, using the algorithm of Ziff, Cummings, and Stell[13,15] which generates perimeters by a random-walk algorithm. Here I considered a virtual lattice of $65\,536^2$ sites, which were divided up into $256^2$ blocks of $256^2$ sites each, and used the data-blocking scheme[13] which assigns memory to a block only when the walk enters it. The memory available, about 2 megabytes, allowed walks of about $2\times10^6$ occupied plus blocked sites to be carried out efficiently. I used an upper cutoff of $1\,048\,576 = 2^{20}$ occupied sites, stopping all walks which did not close by that number of steps. In all the simulations, the walker never wandered more than 5000 steps in any direction, and so the lattice boundary was never seen. 30 600 trials were carried out at $p_c = 0.5928$, requiring $\sim 100$ h of computing time on an Apollo 460 engineering work station. A typical perimeter is shown in Fig. 1.

To find the value of $\tau'$, I kept statistics on the number of perimeters grown to a size $s'$, putting the results in bins in the ranges of 1, 2–3, 4–7, 8–15, . . . , 524 288–1048 575. At $p_c$, the probability of growing to a size $s'$ is $s'n_{s'} \sim (s')^{1-\tau'}$, so that the number of clusters in the bin for sizes $s'$ to $2s'-1$ is proportional to $(s')^{2-\tau'}$. The data for the number in

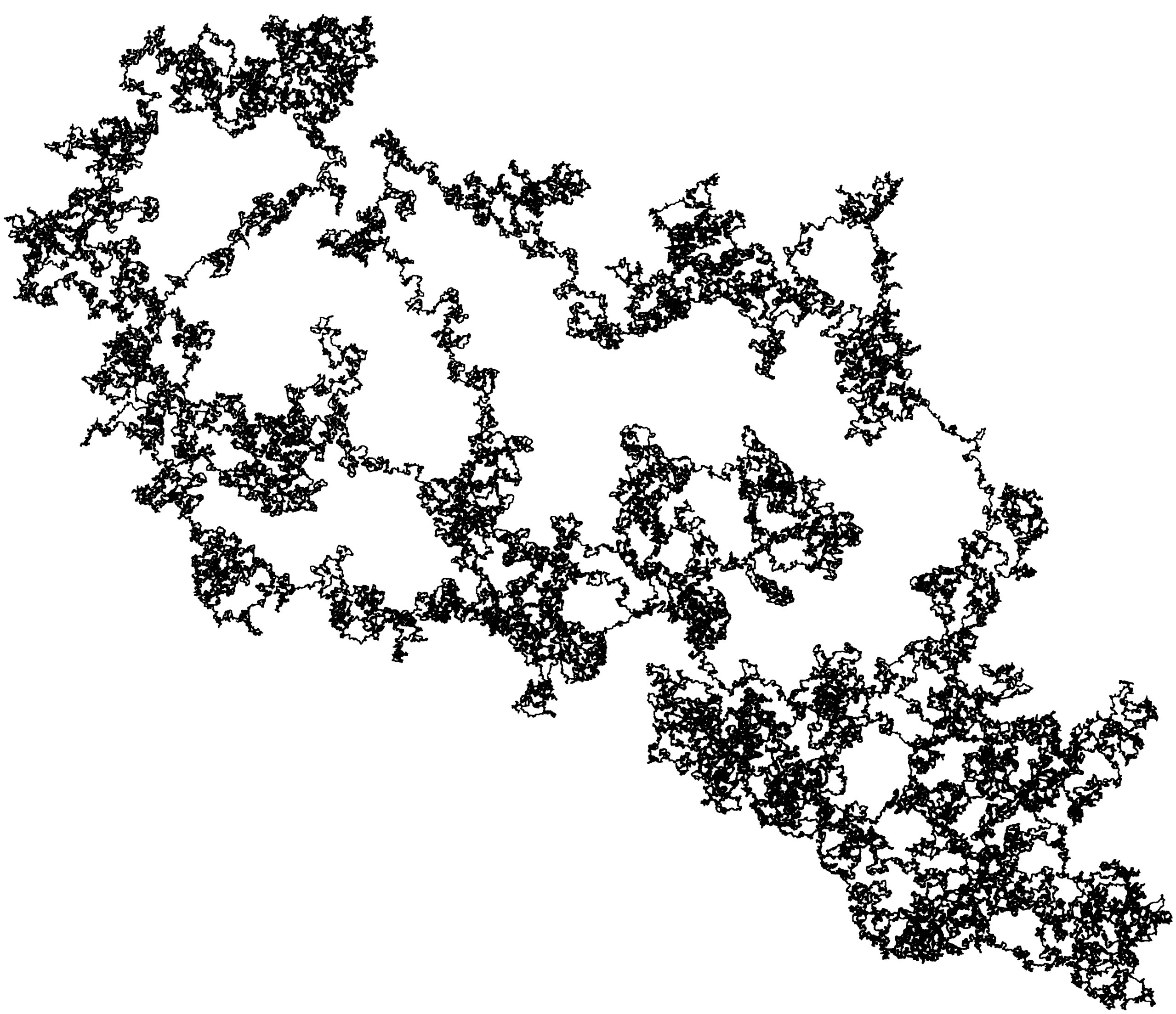

FIG. 1. External perimeter of a percolation cluster generated by the random-walk algorithm, containing 328 061 total sites, 194 464 occupied and the rest blocked. Only the occupied sites are shown. Note the holes over all length scales. The maximum dimensions are 1291 × 1570.

each bin are plotted as squares in Fig. 2 on a log-log plot, and a straight line is drawn through them with the predicted slope $2-\tau' = -\frac{1}{7}$. The data are consistent with this line. To analyze the data it is advantageous to use the upper cumulative distribution

$$N'_{s'} = \sum_{s'}^{\infty} s' n'_{s'} \sim (s')^{2-\tau'}, \tag{8}$$

because this sum includes the walks that hit the cutoff $2^{20}$, and therefore has greater numerical significance. The values of $N'_{s'}$ are plotted in Fig. 2, and the deviation from the line of slope $-\frac{1}{7}$ is imperceptible on this scale. To make that difference visible, $(s')^{1/7}N'_{s'}$ vs $s'$ is also plotted in Fig. 2, and excellent agreement with a horizontal line is noted even on this expanded scale. A linear least-squares fit of the data of $N'_{s'}$ over the range $s' = 2^{10}$ to $2^{20}$ gives $\tau' = 2.143 \pm 0.002$, consistent with the predicted $\frac{15}{7}$.

In the single-perimeter method used here, there are no complicating boundary effects, as in finite-size scaling methods.[3,16–18] The advantage in growing single objects, rather than populating an entire lattice, has been noted for the growth of single percolation clusters.[19] By using an extremely large virtual lattice, one is able to carry out very long walks which make the corrections to scaling insignificant.

For the 4687 incomplete walks of $2^{20}$ steps, I also calculated the fractal dimensionality. The position of every 64th occupied site was recorded on a list (of length 16 385) and pairs from this list were used to find the average distances for points separated by 64, 128, ..., $524\,288 = 2^{19}$ steps. These distances, averaged over all 4687 clusters, are plotted as a function of $s'$ in Fig. 3, a log-log plot, along with a line of slope $\frac{8}{7}$, representing $d'_f = \frac{7}{4}$, since $\langle R^2 \rangle \sim (s')^{2/d'_f}$. Also shown are the data of $\log_2[(s')^{-8/7}\langle R^2\rangle]$, which should fall on a horizontal line if $d'_f = \frac{7}{4}$. A least-squares fit of the data for $s' = 2^{10}$ to $2^{19}$ gives $2/d'_f = 1.142 \pm 0.001$ or $d'_f = 1.751 \pm 0.002$, consistent with the conjectured result, and with Voss's measurement.[5] The last point in Fig. 3, representing points separated by $2^{19}$ steps, falls substantially below the line because some of the walks are about to close when the cutoff is reached.

The results found here are summarized in Table I, along with other known values for perimeter exponents. The exponent $\gamma'$ was tested in a previous simulation[13] in which the perimeter-generating walk was performed at $p < 0.590$ and $p > 0.596$ so that all perimeters closed and the mean size could be determined.

I end with a discussion of the use of the perimeter walk to find the value of $p_c$. The walk is extremely sensitive to $p$, since for walks of $p < p_c$ the average

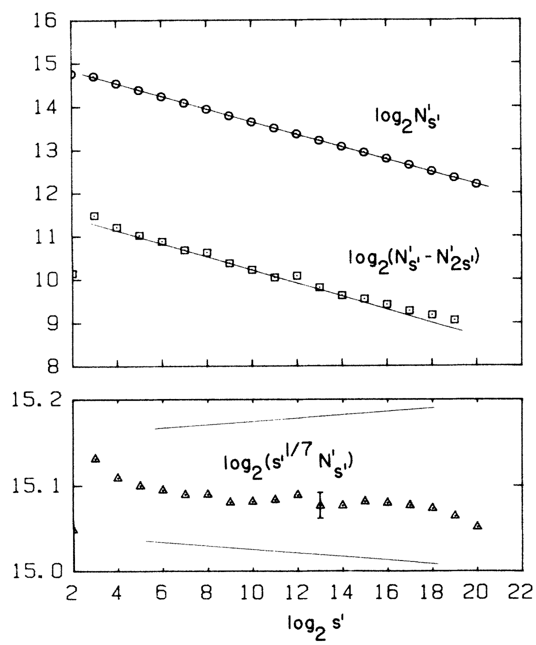


FIG. 2. Measurement of $\tau'$. The lower plot shows the data on a greatly expanded scale, and shows a typical error bar representing the standard deviation of the data. The two oblique lines represent $\tau'$ equal to $\frac{15}{7}+0.002$ and $\frac{15}{7}-0.002$. The error $\pm 0.002$ is the standard deviation of the slopes of consecutive pairs of points in the range $\log_2 s' = 10$ to 20. The deviation for small $s'$ represents finite-size, or correction-to-scaling, effects.

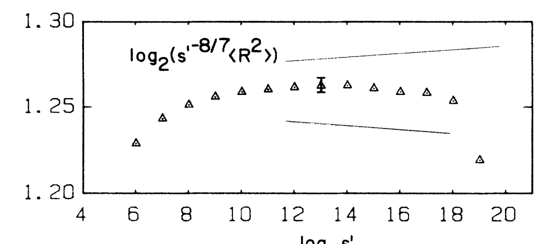


FIG. 3. Measurement of $d'_f$. Again, the lower plot shows the data on a greatly expanded scale, with a typical error bar representing the standard deviation of the data; the correction to scaling behavior is evident for $s < 10$. The two oblique lines show what the slopes would be if $2/d'_f$ were equal to $\frac{8}{7}+0.001$ and $\frac{8}{7}-0.001$, or $d'_f = \frac{7}{4} \pm 0.002$.

TABLE II. The number of external and internal perimeters with $n_{occ}$ in the range $2^{16}$ to $2^{20}-1$. The error bounds on the ratio represent the statistical error ( $\pm N^{-1/2}$) on the number of measurements. These results imply $p_c = 0.592\,75 \pm 0.07(\Delta p/\Delta \text{ratio}) = 0.592\,75 \pm 0.000\,03$.

| $p$ | Trials | External | Internal | Ratio |
|---|---|---|---|---|
| 0.5926 | 5000 | 236 | 153 | 1.54 ±0.23 |
| 0.5927 | 10000 | 386 | 333 | 1.16 ±0.12 |
| 0.59275 | 25000 | 938 | 945 | 0.99 ±0.07 |
| 0.5928 | 10000 | 337 | 372 | 0.91 ±0.10 |
| 0.5929 | 9000 | 260 | 420 | 0.63 ±0.07 |
| 0.5930 | 5000 | 137 | 257 | 0.53 ±0.08 |

length of external perimeters is more than 100 times greater than the average of internal perimeters, while for $p > p_c$ it is the other way around[13]:

$$\langle s'\rangle_{-}^{\text{ext}} \sim \langle s'\rangle_{+}^{\text{int}} \sim A\,|p-p_c|^{-2},$$
$$\langle s'\rangle_{-}^{\text{int}} \sim \langle s'\rangle_{+}^{\text{ext}} \sim B\,|p-p_c|^{-2}, \quad (9)$$

with $A = 0.5$ and $B = 0.004$, where the subscripts + and − represent above and below $p_c$, respectively. Close to $p_c$, the difference between internal and external perimeters should become apparent when the walk exceeds about $B|p-p_c|^{-2}$ steps, or $10^6$ steps for $|p-p_c| = 0.000\,06$. Thus, the walks of $2^{20}$ steps considered here should be easily sensitive to this difference in $p$.

I thus carried out simulations at $p = 0.5926, \ldots,$ 0.5930 keeping track of the direction of closing as well as the length for those perimeters that closed before the cutoff of $n_{occ} = 2^{20}$. The number of trials for each value of $p$ and the number of internal and external perimeters in the range $2^{16}$ to $2^{20}-1$ are shown in Table II. From these data, I conclude that $p_c$ is 0.592 75 with a statistical error of ±0.000 03 as explained in Table I. This value is consistent with the recent result of Gebele,[20] $p_c = 0.592\,77 \pm 0.000\,05$. Gebele's calculation required ~23 h of computer time on a CDC Cyber 76 computer, which is hundreds of times faster than the Apollo computer used in this work. The six runs here required about 160 h on the Apollo. The gain in efficiency can be traced to the large difference between $A$ and $B$ in (9), which makes the perimeter walk very sensitive to $p$.

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*Note added.*—Very recently, Rosso, Gouyet, and Sapoval[21] have proposed that $p_c = 0.592\,802 \pm 0.000\,010$. The result and the result found here differ by slightly more than the combined error bars.

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