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Limiting Quasienergy Statistics for Simple Quantum Systems

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The limiting distribution $P(s)$ of quasienergy-level spacings is investigated for a quantum model which is stochastic in the classical limit. The numerical results show that in the case of a strong perturbation, the distribution $P(s)$ corresponds closely to the Wigner-Dyson distribution. The relations between the limiting distribution $P(s)$ and the symmetries of both the unperturbed and perturbed motions are independently investigated.

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It has been shown[1] that there is a correspondence between the stochastic behavior of a classical system and the irregularity of the energy spectra of the corresponding quantum system. Energy-level fluctuations are characterized by many quantities (see, for example, Brody *et al.*[2]) but the most commonly used is the spacing distribution $P(s)$ of nearest-neighbor levels. This distribution is the main feature of the Wigner-Dyson statistical description[3,4] of such complex systems as heavy nuclei and atoms. In its simplest form, this distribution is given by[4]

$$P(s) = As^{\beta}e^{-Bs^2}, \tag{1}$$

where A and B are normalizing constants, and β is a parameter characterizing the repulsion between neighboring energy levels. This level repulsion, which was discussed as far back as the 1950's,[5,6] is typical for quantum systems with classically chaotic behavior.[7,8] The value of β depends on the symmetry of the unitary random matrices characterizing the system under consideration.[3,4]

The distribution (1) with linear repulsion, $\beta=1$, is, in general, confirmed by experimental data for heavy nuclei[9] and complex atoms.[10] There are also a number of numerical experiments with simple quantum systems.[7,8,11] Recently the distribution (1) with $\beta=2$ was found, numerically, by Seligman and Verbaarschot[2] for a model with two degrees of freedom. However, the data from most of these experiments are not statistically conclusive, and it is not possible to assert with confidence that they agree exactly with the theoretical dependence (1). The only exception is the numerical result on the Sinai billiard,[13] for which the correspondence is remarkably good with a high χ^2 confidence level.

In this paper the Wigner-Dyson distribution (1) is shown to be also typical of nonconservative systems under a time-dependent periodic perturbation. Such a distribution (for quasienergy levels) appears when the perturbation is strong, and results from the delocalization and ergodicity of the quasienergy eigenfunctions throughout the momentum space of the system. In this limiting case, statistical properties only depend on the symmetry of the given system.

Let us consider, as an example, a quantum rotator which is subjected to periodic kicks. Its Hamiltonian has the form (in dimensionless units)

$$H = -\frac{1}{2}\frac{\partial^2}{\partial\theta^2} + V(\theta)\delta_T(t); \quad V(\theta) = k\cos\theta. \tag{2}$$

Here $\delta_T(t)$ is a periodic delta function with period T and $V(\theta)$ is the perturbation potential with strength parameter k. We are dealing here with a general case of nonresonant values, where $T/4\pi \neq p/q$ (p,q are integers). The special case of quantum resonance, when $T/4\pi = p/q$, was considered by Izrailev and Shepelyansky.[14] Numerical experiments[15–18] with the model (2) have shown that the stochasticity associated with the corresponding classical "standard mapping" is substantially reduced by an effect called "quantum suppression." It is clear now[16,19] that this suppression results from a localization of the quasienergy eigenfunctions in momentum space.

It has been shown[20] that the model (2) can be transformed into another model which relates to the problem of disordered one-dimensional systems.[21] For the case of small perturbation, $k \lesssim 1$, exponential localization has been numerically observed[20] in a discrete configuration space. Such a quantum localization is analogous to the well known Anderson localization[22] in a random potential and results in a restriction of the diffusion for any initially localized state in the configuration space. Correspondingly, for the given model (2), the localization of the quasienergy eigenfunction leads to the restriction of the diffusion in momentum space and, as a result, to a limitation of the energy growth.

For conservative systems, the behavior of the quantum system is expected to resemble most closely the stochastic classical motion when all of the eigenfunctions in the Wigner representation are ergodic and, therefore, extend throughout the phase space of the system.[23] This is just the case of "dispersed billiards,"[13,24,25] for which the law (1) with $\beta = 1$ has been confirmed numerically. It is interesting to note that for the "stadion,"[25] unlike the Sinai billiard,[13] there is a noticeable percentage of eigenfunctions which are not fully ergodic. This is probably the main reason for the discrepancy between the distribution $P(s)$ (see Ref. 25) and the theoretical prediction (1). There, the χ^2 confidence level is very poor because of high values of $P(s)$ at large s.

By analogy with conservative systems it is of interest to investigate the distribution of spacings $P(s)$ of nearest-neighbor quasienergy levels. For the quantum rotator (2) with localization, we expect the repulsion of quasienergies to be absent. This conclusion is associated with the fact that the probability for the overlap of quasienergy eigenfunctions in the unbounded momentum space goes to zero (see also Feingold *et al.*[26])

To investigate the statistics of the limiting quasienergy spectrum (when all eigenfunctions are extended) it is convenient to modify our model (2) in such a way that the momentum space becomes periodic. This means that the phase space of the corresponding classical system becomes a two-dimensional torus. It is known that this does not change statistical properties of the motion.[18] An essential feature of this new quantum model is that the number of quasienergy levels is finite.

We first write an exact solution of the Schrödinger equation for the model (2) after one perturbation period T:

$$\psi(\theta, t+T) = e^{-iV(\theta)} e^{-iH_0 T} \psi(\theta, t). \tag{3}$$

The equation for the quasienergy eigenfunction $\phi_\epsilon(\theta)$ can be written from (3) as[14]

$$e^{i\epsilon T}\phi_\epsilon(\theta) = \{\exp(-ik\cos\theta)\exp(-\tfrac{1}{2}iT\,\partial^2/\partial\theta^2)\}\phi_\epsilon(\theta). \tag{4}$$

It is seen that the quasienergies ϵ are generally determined (mod $2\pi/T$) by the eigenvalues $\lambda = \exp(i\epsilon T)$ of some infinite unitary matrix S. The matrix elements $S_{\alpha\delta}$ of the operator $\hat{S} = \hat{B}\hat{G}$ [where $\hat{B} = \exp(-ik\cos\theta)$ and $\hat{G} = \exp(-\frac{1}{2}iT\,\partial^2/\partial\theta^2)$], in a basis with respect to which the operator $\hat{B}$ is diagonal, are defined by

$$S_{\alpha\delta} = e^{-ik\cos\theta_\alpha} \sum_{l=-\infty}^{\infty} e^{i(T/2)l^2} e^{-il(\theta_\alpha - \theta_\delta)}. \tag{5}$$

All of the eigenfunctions ϕ_ϵ and eigenvalues λ of this infinite matrix S depend on the continuous parameters $\theta_\alpha, \theta_\delta$ lying in the interval $(0, 2\pi)$ and give the total solution of the model (2) with an unbounded momentum space. In accordance with this, it can be shown that, for a model with a finite number N of quasienergy levels, the matrix S is determined by

$$S_{mn} = B_m G_{mn} = \frac{1}{2N_1+1}\exp\left[-ik\cos\left(\frac{2\pi}{N}m + \theta_0\right)\right] \sum_{l=-N_1}^{N_1} \exp\left(i\frac{T}{2}l^2\right)\exp\left[-i\frac{2\pi}{N}l(m-n)\right]. \tag{6}$$

Here the parameter θ_0 ranges from 0 to $2\pi/N$.

It is clear that the model (6) can be regarded as a discrete approximation to the continuous system (2), (5). The phase space of this system is discrete both in phase θ and in momentum l. In order to keep the symmetry of the perturbation $V(\theta) = V(-\theta)$ the free parameter θ_0 has to be either $\theta_0 = 0$ or $\theta_0 = \pi/N$. Moreover, the value of N_1 in (6) should be $N_1 = (N-1)/2$ with $N-1$ an even integer. Then the expression (6) in the limit $N \to \infty$ turns into (5) and for rational values of $T/4\pi = p/q$ (p and q even, $q = N$) coincides with the matrix elements of the system in quantum resonance.[14]

The dependence of $P(s)$ on k and T was investigated numerically elsewhere[27] for the model (6). In particular, it was found that as k increases, the distribution $P(s)$ (with $\mathcal{K}=kT=5$) changes from a Poisson-type distribution (small k) to the Wigner-Dyson distribution (1) with $\beta=2$. The limiting ($k \gg N \gg 1$) distribution $P(s)$ turns out to be very close to (1) with a high confidence level. The quadratic law of repulsion ($\beta=2$) is related to the fact that in Ref. 27 the general case was considered in which there are no symmetries in the matrix S. Specifically, the model there had a nonsymmetric sum (from 1 to N) in (6) and arbitrary values of θ_0. It corresponds to the case in which both the unperturbed motion and the perturbation are noninvariant under the transformation $\theta \rightarrow -\theta$. Also, the motion of such a system is not time invariant as a result of the absence of any symmetry in the matrix S.

Now we focus our attention on the relation between the limiting distribution $P(s)$ and the symmetry of the model under consideration. Let us consider, first, the model (6) with the same symmetries as the original system (2). Note that for (2) the conditions $H_0(\theta)=H_0(-\theta)$ and $V(\theta)=V(-\theta)$ are satisfied simultaneously. This implies that an additional integral exists which is a parity. Correspondingly, the quasienergy eigenfunctions $\phi_\epsilon(\theta)$ have to be either even or odd: $\phi_\epsilon(\theta)=\pm\phi_\epsilon(-\theta)$. Hence, the quasienergy spectrum should be treated for these two sets of eigenfunctions independently.

Figure 1(a) shows the distribution $P(s)$ which was obtained numerically for this case. To improve the statistics, ten matrices of size $N=99$ were considered for different values in the neighborhood of $k \approx 20\,000$. Because of the strong sensitivity of the eigenvalues λ to small changes in the parameter k (with step $\Delta k=1$), these ten sequences of quasienergy levels can be regarded as mutually independent. Also, the distributions $P(s)$ for the even and odd eigenfunctions have been summed so that the total number of quasienergy levels is equal to $M=990$. A very large value of k was taken to ensure that all the eigenfunctions were completely extended. From Fig. 1(a) a rather good correspondence is seen between the experimental distribution $P(s)$ and the theoretical law (1) for $\beta=1$. The χ^2 value is equal to 14.8 for 23 degrees of freedom with a confidence level $w \approx 10\%$. It can be proven that this model is not only space invariant but time invariant as well. Correspondingly, the matrix S has two special symmetries which reduce the total number of independent matrix elements by a factor of $\frac{1}{4}$. It makes clear why the distribution $P(s)$ for very large k is the same as that for the matrix corresponding to a Gaussian orthogonal ensemble of random unitary matrices.[2-4]

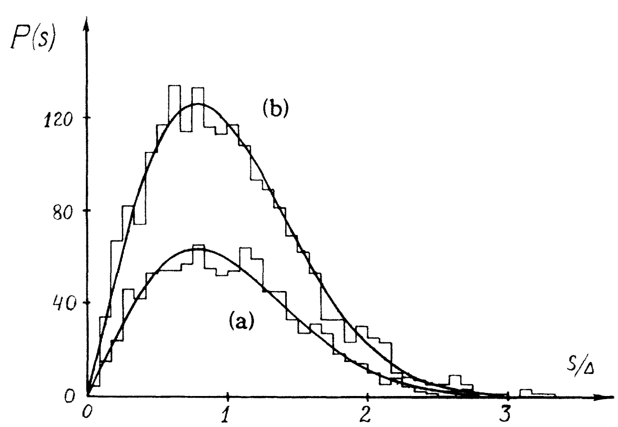

FIG. 1. Distribution $P(s)$ with s a distance over the unit circle between the nearest-neighbor eigenvalues λ; $T=\frac{1}{3}\sqrt{3}$, $\theta_0=\pi/N$, $\Delta=2\pi/N$. The solid curves give the analytic dependence (1) with $\beta=1$, and the histograms show the numerical data: (a) $H_0=-\frac{1}{2}\,\partial^2/\partial\theta^2$, $N=99$, $M=990$, $\chi^2_{23}\approx 14.8$, $w\approx 10\%$; (b) $H_0=-\frac{1}{2}\,\partial^2/\partial\theta^2+(i/\sqrt{2})\,\partial/\partial\theta$, $N=199$, $M=1990$, $\chi^2_{27}\approx 34.7$, $w\approx 15\%$.

The distribution $P(s)$ with linear repulsion appears also for the model in which only the unperturbed motion is space invariant $[H_0(\theta)=H_0(-\theta); V(\theta) \neq V(-\theta)]$. It is easy to construct such a model if we assume $\theta_0 \neq 0$ or π/N in (6). In this case, the parity is not conserved but the system remains time invariant. The corresponding symmetry of the matrix S halves the number of independent matrix elements. It can then be assumed that, for large k, this matrix S corresponds to a Gaussian orthogonal ensemble. The numerical data give $\chi^2 \approx 35.2$ for 27 degrees of freedom with $w \approx 14\%$ (for $N=199$, $M=1990$, $k\approx 20\,000$, $T=\frac{1}{3}\sqrt{3}$, $\theta_0=\pi/20N$).

An interesting result has been obtained for this system with $V(\theta)=V(-\theta)$ but $H_0(\theta)\neq H_0(-\theta)$. In this case, the model (6) has been slightly modified to correspond to the system (2) with a complex Hamiltonian $H_0=-\frac{1}{2}\,\partial^2/\partial\theta^2+i\gamma\,\partial/\partial\theta$ but with the same perturbation $V(\theta)=k\cos\theta$. Such a Hamiltonian is characteristic of a magnetic field (see also Ref. 12). Figure 1(b) shows that this case gives again the distribution (1) with $\beta=1$. Note that here both the space invariance and the time-reversal invariance are destroyed [the matrix

$$G_{mn}=\sum_{-N_1}^{N_1}\exp\left(i\frac{T}{2}l^2-i\gamma l\right)\exp\left(i2\pi l\frac{m-n}{N}\right)$$

has nonsymmetrical form]. Nevertheless, the matrix S proves again to have a special symmetry and thus, again, there are only $N/2$ independent complex matrix elements. This symmetry corresponds to the conservation of TP invariance of the system under the

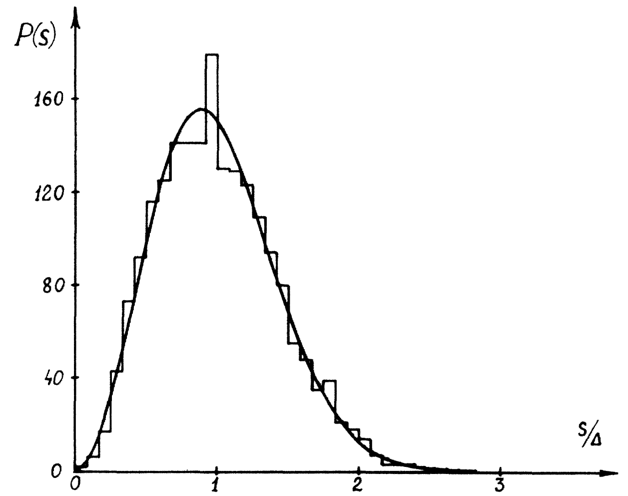

FIG. 2. The same as Fig. 1 for $\theta_0 \approx 8\times 10^{-4}$, $H_0 = -\frac{1}{2}\,\partial^2/\partial\theta^2 + (i/\sqrt{2})\,\partial/\partial\theta$, $N=199$, $M=1990$, $\chi^2_{27} \approx 18.9$, $w \approx 65\%$. The smooth curve shows the analytical dependence (1) with $\beta = 2$.

transformation $t \to -t$ together with $\theta \to -\theta$.

Finally, for a system in which the symmetries in both H_0 and V are broken $[V(\theta) \neq V(-\theta)$ and $H_0 = -\frac{1}{2}\,\partial^2/\partial\theta^2 + i\gamma\,\partial/\partial\theta]$ the distribution (1) is obtained with quadratic ($\beta = 2$) repulsion (Fig. 2). In this case, no additional symmetries exist and the matrix S is of the general form with N independent complex matrix elements. It looks like the matrix corresponding to a Gaussian unitary ensemble.[2-4]

In conclusion, it should be pointed out that the matrix S, describing the motion of our model with a finite number of quasienergy levels, unlike that of the Wigner-Dyson theory, depends on the parameters k and T and, in the general case, is not a random matrix. Only in the case of very large k (and $T \sim 1$) does the distribution $P(s)$ resemble that of the random unitary matrices. In the other limit (small $k \ll 1$) one might expect (see Berry and Tabor[28]) a Poisson distribution. However, this question should be considered more carefully, because there exist numerical experiments[29] with conservative integrable systems which do not give exact correspondence to the Poisson law.

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