Hadron Dynamics in the Three-Flavor Skyrme Model

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We present the three-flavor formalism for scattering of pseudoscalar mesons from baryons in the class of models in which the baryon is viewed as a "hedgehog" soliton in the meson field. To test this formalism, we apply it to πN scattering in the Skyrme model. The result, as compared with the two-flavor Skyrme model, is an overall improvement in agreement with experiment.

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Despite universal acceptance of the quark model, recent times have seen a revival of interest in Skyrme's imaginative treatment of the baryon as a soliton in the field of pions.[1] The soliton picture of the baryon can be motivated from a consideration of QCD in the limit in which N_c, the number of colors, is taken to infinity.[2,3] The popularity of the Skyrme model in particular is largely due to the ease with which it permits moderately accurate calculations of a wide variety of hadronic properties.[4,5] The starting point is the modified nonlinear σ model:

$$\mathscr{L}=(f_\pi^2/16)\mathrm{Tr}\partial_\mu U\partial^\mu U^\dagger+(1/32e^2)\mathrm{Tr}[(\partial_\mu U)U^\dagger,(\partial_\nu U)U^\dagger]^2,\quad U\in\mathrm{SU}(2), \tag{1}$$

which admits a hedgehog soliton solution ("skyrmion")

$$U_0=\exp\{iF(r)\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}\}, \tag{2}$$

where F solves a nonlinear variational equation. Of course, any rotation AU_0A^{-1} is an equally acceptable solution; indeed, the proper identification of (2) with a nucleon or Δ requires the quantization of the "collective coordinates" A.[4]

Pions can readily be added to the model[6] by consideration of fluctuations about the skyrmion[7]:

$$U_0\rightarrow\exp\{i[F(r)\hat{\mathbf{r}}+2\boldsymbol{\pi}(\mathbf{x},t)/f_\pi]\cdot\boldsymbol{\sigma}\}. \tag{3}$$

In so doing one can explicitly calculate the S matrix for πN scattering and in the process obtain the spectrum of N and Δ resonances, with surprisingly good results: rough qualitative agreement for most partial-wave amplitudes,[8,9] and 8% agreement on average with experimental resonance masses.[8] In addition, using only the hedgehog structure of the skyrmion, one can derive model-independent linear relations between experimental partial wave amplitudes for $\pi N\rightarrow\pi N$ and $\pi N\rightarrow\pi\Delta$, with similar success.[9,10]

In retrospect, these successes of the model are probably due in part to the fact that, to leading order, the πN phase shifts are independent of N_c.[2] Indeed, it appears that calculations in the Skyrme model have the best chance of quantitative success if they are formulated in such a way that the leading dependence on N_c explicitly cancels out. For example, while Adkins *et al.* obtained values for μ_p and μ_n that were 30% lower than experiment, the *ratio* μ_p/μ_n is accurate to better than 3%. Similarly, while their values for $f_\pi\sim N_c^{1/2}$ and $g_A\sim N_c$ are in poor agreement with experiment, the quantity $f_\pi^2/g_A\sim N_c^0$ turns out to be likewise accurate to 3%.[11]

It is important to consider the effect of incorporating additional low-lying mesons into (1). In this paper we begin a study of the extension of meson-skyrmion scattering to the case of three light flavors. The crucial first question addressed here is whether the two-flavor successes of the model obtained in Refs. 8 and 9 survive this extension. In fact, we shall find a modest overall improvement in agreement with experiment. Elastic πN scattering is a particularly rigorous proving ground for the three-flavor formalism, because of the reliability of the experimental phase shifts.[12] In a later paper, we will discuss the comparison of theory with experiment for channels involving strange particles. For purposes of simplicity, we shall work here in the chiral limit of unbroken $\mathrm{SU}(3)_L\otimes\mathrm{SU}(3)_R$, restricting ourselves, as in Refs. 8 and 10, to a leading-order analysis in $1/N_c$.[13]

We begin with a sketch of the general formalism for bouncing bosons from baryons in the three-flavor Skyrme model; a detailed derivation will be presented elsewhere.[14] One should keep in mind that, apart from Eq. (6) below, the development of this section is valid, to leading order in $1/N_c$, for *any* three-flavor chiral soliton model in which the skyrmion is a hedgehog.

The usual starting point for three-flavor chiral soliton models[3,15] consists of embedding of the skyrmion (2) in the upper left of a 3×3 matrix. It is fruitful to forget about the baryon's collective coordinates at first, and to concentrate instead on the simplified problem of a pseudoscalar-octet meson ϕ^a scattering from an *unrotated* skyrmion; we shall refer to this as *reduced* scattering. The ϕ's are naturally incorporated into the Lagrangean by letting [cf. Eq. (3)]

$$U_0 = \exp\left\{iF(r)\sum_{i=1}^{3}\hat{r}^i\lambda^i\right\} \rightarrow \exp\left\{iF(r)\sum_{i=1}^{3}\hat{r}^i\lambda^i + \frac{2i}{f_\pi}\sum_{a=1}^{8}\phi^a\lambda^a\right\} \tag{4}$$

with λ^a the Gell-Mann matrices. For the case of three or more flavors, the action must be augmented by the addition of the Wess-Zumino (WZ) term, which correctly reproduces the flavor-current anomalies of the strong interactions.[3,16] The complete action is then expanded about the skyrmion to quadratic order in the ϕ's. In particular, the WZ term makes a contribution

$$(iN_c/4\pi^2)\int d^4x\,[F'(\sin^2F)/F^2r^2](1-\cos F)(K^-\dot{K}^+ + \bar{K}^0\dot{K}^0) \tag{5}$$

to the action.[17] $O(\phi^3)$ terms are ignored in our lowest-order treatment, as they are suppressed by powers of $1/f_\pi \sim 1/\sqrt{N}$. The result is a set of linear second-order Euler-Lagrange equations for the ϕ's.

As in the two-flavor procedure, we can obtain an effective radial problem by expanding ϕ^a in eigenstates of the symmetries of the unrotated skyrmion; in this case $(\mathbf{K}^2, K_z, Y)$. Here $\mathbf{K}$ (not to be confused with the kaon!) is the vectorial sum $\mathbf{I}+\mathbf{L}$ of the meson's isospin and angular momentum, and Y is its hypercharge. Each effective radial problem is then characterized by a "reduced" amplitude $s_{KL'L}^{\{IY\}}(\omega)$, with L and L' henceforth denoting the initial and final meson partial waves. Specifically, the meson fluctuations decompose into the following noninteracting sectors.

(a) There are the fluctuations in the pion directions: $\{I,Y\}=\{1,0\}$. These are expanded into radial functions $\phi_{KK_zL}(r)e^{i\omega t}$, $K=\{L-1,L,L+1\}$, summed against the vector spherical harmonics $\Pi_L^{KK_z}(\Omega)$ familiar from the two-flavor analysis. The resulting equations of motion for the ϕ's can be integrated numerically out to large values of r, where the effect of the skyrmion's tail is negligible and the theory is one of free mesons. In this region, the ϕ's can be fitted to a sum of incoming and outgoing Bessel functions, and the phase shifts extracted in the usual manner. Exponentiation yields the reduced S-matrix elements $s_{KL'L}^{\{10\}}$; a moment's thought will confirm that these are identically the same as for the two-flavor case, as depicted in Ref. 6 and by Walliser and Eckart.[18]

(b) There are the fluctuations in the η direction, $\{I,Y\}=\{0,0\}$, expanded in the usual spherical harmonics. The result of fluctuations in this direction is just free field theory: $s_{KL'L}^{\{00\}}(\omega) \equiv \delta_{KL}\delta_{LL'}$.

(c) In analogy to (a), there are fluctuations $\psi_{KK_zL}(r)e^{i\omega t}$, $K=L\pm\frac{1}{2}$, in the direction of the kaon or antikaon doublet, summed against the "spinor spherical harmonics"

$$\mathscr{Y}_L^{KK_z}(\Omega) = \begin{pmatrix} \langle L\tfrac{1}{2}, K_z-\tfrac{1}{2}, \tfrac{1}{2}|KK_z\rangle\, Y_{L,K_z-1/2}(\Omega) \\ \langle L\tfrac{1}{2}, K_z+\tfrac{1}{2}, -\tfrac{1}{2}|KK_z\rangle\, Y_{L,K_z+1/2}(\Omega)\end{pmatrix}.$$

By parity, $\psi_{KK_z,K-1/2}$ cannot scatter into $\psi_{KK_z,K+1/2}$, so that L must equal L'. The effective radial equation of motion for fluctuations in this direction turns out to be

$$\begin{aligned}(\tilde{r}^2+2\sin^2F)\frac{d^2}{d\tilde{r}^2}(\psi_{KK_zL}/F) + \left[2\tilde{r} - 4F'\sin F + F'(1+\cos F)\left(\frac{\tilde{r}^2}{\sin F}+6\sin F\right)\right]\frac{d}{d\tilde{r}}(\psi_{KK_zL}/F) \\ + \Bigg\{[L(L+1)-K(K+1)-\tfrac{5}{4}]\left\{2+3F''\sin F+\frac{5\sin^2F}{\tilde{r}^2}-(F')^2-(1+\cos F)\left[1+\frac{4\sin^2F}{\tilde{r}^2}-2(F')^2\right]\right\} \\ -(L-1)(L+2)\left[1+\frac{\sin^2F}{\tilde{r}^2}+(F')^2\right]+\omega^2[\tilde{r}^2+2\sin^2F+\tilde{r}^2(R')^2] \pm \frac{\omega N_c}{\pi^2}F'\sin^2F\Bigg\}(\psi_{KK_zL}/F)=0.\end{aligned} \tag{6}$$

The final term in (6) represents the effect of the WZ term, the + and − signs referring to the antikaon and kaon fluctuations, respectively (cf. Calland and Klebanov[19]). Numerically, the contribution of the WZ term turns out to be extremely small, so that $s_{KLL'}^{\{1/2,1\}} \cong s_{KLL'}^{\{1/2,-1\}}$.

We have just given the complete recipe for constructing the reduced S matrix that characterizes the scattering of a pseudoscalar-octet meson from an *unrotated skyrmion. The conserved quantities for this unphysical process are the* sum $\mathbf{K}=\mathbf{I}+\mathbf{L}$ of the meson's isospin and angular momentum, and the meson's hypercharge Y. Of course, these are not preserved in *physical* three-flavor meson-baryon scattering, for which the conserved quantities are the total meson-baryon angular momentum $\mathbf{J}$, and the total $SU(3)_{\text{flavor}}$ quantum numbers $\{R_{\text{tot}}, \gamma, I_{\text{tot}}, I_{z\text{tot}}, Y_{\text{tot}}\}$.[20] Pleasingly, these conservation laws emerge naturally from the skyrmion formalism once the collective coordinate structure of the baryons is properly taken into account.[14] Other physically relevant (albeit not necessarily conserved) quantum numbers are the meson partial wave L, and the spin s and flavor representation $\mathbf{R}$ of the baryon [i.e., $(s,\mathbf{R})=(\frac{1}{2},\mathbf{8})$ or $(\frac{3}{2},\mathbf{10})$]. As in the two-flavor case,[8-10] the S matrix characterizing physical scattering in three-flavor skyrmion models can be expressed as a linear combination of the reduced amplitudes described above[14]:

$$S(\{Ls\mathbf{R}R_{\rm tot}\gamma I_{\rm tot}I_{z\,\rm tot}Y_{\rm tot}\mathbf{J}\}\to\{L's'\mathbf{R}'R'_{\rm tot}\gamma' I'_{\rm tot}I'_{z\,\rm tot}Y'_{\rm tot}\mathbf{J}'\})$$
$$=\delta_{R_{\rm tot}R'_{\rm tot}}\delta_{I_{\rm tot}I'_{\rm tot}}\delta_{I_{z\rm tot}I'_{z\rm tot}}\delta_{Y_{\rm tot}Y'_{\rm tot}}\delta_{JJ'}\delta_{J_zJ_z'}(-1)^{s'-s}\frac{(\dim\mathbf{R}\times\dim\mathbf{R}')^{1/2}}{\dim R_{\rm tot}}$$
$$\times\sum_{\{IY\}}\sum_i\sum_K(2i+1)(2K+1)\begin{Bmatrix}K&i&J\\ s'&L'&I\end{Bmatrix}\begin{Bmatrix}K&i&J\\ s&L&I\end{Bmatrix}\left(\begin{array}{c|cc}R_{\rm tot}\gamma'&\mathbf{R}'&8\\ i,1+Y&s'1&IY\end{array}\right)\left(\begin{array}{cc|c}\mathbf{R}&8&R_{\rm tot}\gamma\\ s1&IY&i,1+Y\end{array}\right)s^{\{IY\}}_{KL'L}.$$

The quantities in braces and parentheses are $6j$ symbols and SU(3) isoscalar factors,[21] respectively. The summation indices i and K run over all values allowed by the triangle inequalities implicit in the $6j$ symbols, and $\{IY\}$ is summed over $\{10\}$, $\{00\}$, and $\{\frac{1}{2},\pm1\}$. The long string of Kronecker δ's expresses the conservation of total angular momentum and $SU(3)_{\rm flavor}$, as promised.

We are now in a position to compare three-flavor meson-baryon scattering in the Skyrme model to nature. We will focus in the present paper on the familiar process $\pi N\to\pi N$. Note that, according to Eq. (7), there are contributions to this process from the "strange" reduced amplitudes with $\{IY\}\neq\{10\}$.

Elastic πN scattering was the subject of exhaustive analysis in the context of the two-flavor Skyrme model. The result was close agreement with the observed spectrum of nucleon and Δ resonances.[8] The four F-wave amplitudes were particularly closely reproduced.[8,9] There was substantial disagreement in the P_{11}, P_{33}, and S_{31} channels[22] which we attributed to mixing with the rotational and translational zero modes of the underlying soliton. The higher partial waves were in good accord with experiment, the main source of discrepancies being the overly elastic nature of the Skyrme model amplitudes.

Figure 1 depicts the two- and three-flavor H-wave amplitudes in the Skyrme model as compared to nature. Clearly the size of the amplitude has moved into closer agreement with experiment. The same pattern holds for most partial waves, and is due to the opening of additional inelastic channels such as ΣK in the three-flavor approach. Note that the three-flavor Skyrme model does just as good a job as the two-flavor model in mimicking the "big-small-small-big" pattern which characterizes the experimental curves for nearly all partial waves. Specifically, the amplitudes with $\{I_{\rm tot},J_{\rm tot}\}=\{\frac{1}{2},L-\frac{1}{2}\}$ or $\{\frac{3}{2},L+\frac{1}{2}\}$ are marked by much greater excursions through the unitarity circle than those with $\{I_{\rm tot},J_{\rm tot}\}=\{\frac{1}{2},L+\frac{1}{2}\}$ or $\{\frac{3}{2},L-\frac{1}{2}\}$. We should emphasize, however, that the poor agreement in the P_{11}, P_{33}, and S_{31} channels is *not* improved; improvement in these channels must await a higher-order $1/N_c$ analysis.[8,10]

A particularly intriguing modification of the two-flavor results occurs in the F_{15} and F_{37} channels (Fig. 2). The dominant peaks in these graphs indicate

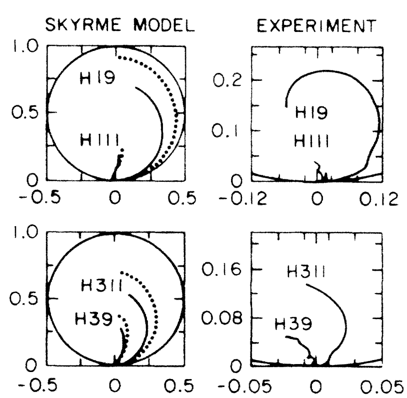

FIG. 1. The four independent H-wave T matrices for the two- and three-flavor Skyrme models (dotted and solid lines, respectively) compared with experiment.

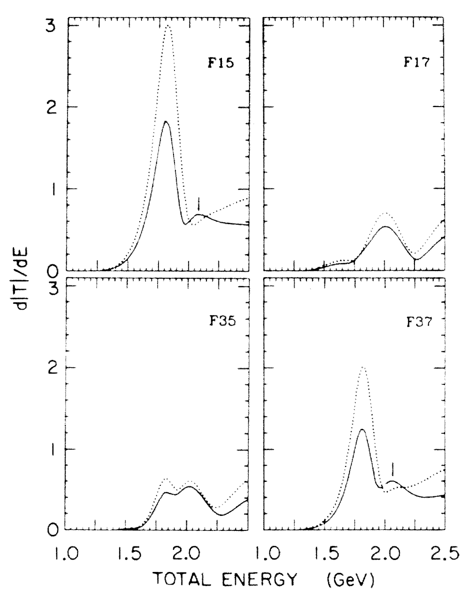

FIG. 2. Speed diagrams for the four F-wave amplitudes in the two- and three-flavor Skyrme models (dotted and solid lines, respectively).

Skyrme-model resonances at roughly 1820 MeV,[23] in reasonable accord with the four-star[24] $F_{15}(1684)$ and $F_{37}(1913)$ states found in nature. The interesting new feature is the emergence in the three-flavor model of additional (weak) resonances at 2060 MeV, in plausible correspondence with the experimental one-star $F_{15}(1882)$ and two-star $F_{37}(2425)$. Suggestively, no such second peak emerges from the Skyrme model in the F_{17} channel, where in nature no second resonance is observed. The F_{35} amplitude in both the two- and three-flavor models is characterized by two overlapping resonances at 1830 and 2030 MeV, although the experimental situation here is somewhat unclear: Although the traditional assignment is to a single broad resonance centered at 1905 MeV with the caveat that "there might be additional structure,"[12] the experimental speed graph seems to reveal two nearby peaks,[8] and recent work does in fact point to two closely-spaced resonances.[25] Curiously, a similar splitting of the F_{35} is predicted by the quark model.[26]

Finally, it should be emphasized that the values of the resonance masses are hardly affected by the inclusion of strangeness, as is exemplified in Fig. 2. In particular, the 8% "best-fit" agreement with experiment found in Ref. 8 continues to hold. Overall, the inclusion of a third light flavor improves the agreement between the Skyrme model and experiment for $\pi N \rightarrow \pi N$.

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