Scaling and Critical Slowing Down in Random-Field Ising Systems

Daniel S. Fisher
AT&T Bell Laboratories, Murray Hill, New Jersey 07974
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A simple scaling description of the ordering transition in random-field Ising systems is developed and supported by renormalization-group arguments in terms of a zero-temperature critical fixed point. The main prediction is that the characteristic relaxation time τ will diverge extremely rapidly as the critical point is approached: $\tau \sim \exp(\xi^{\theta})$ with ξ the correlation length and θ the "violation of hyperscaling" exponent $(d-\theta)\nu = 2-\alpha$. Recent experiments which exhibit onset of hysteresis in a very narrow temperature range are discussed.

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The nature of the ordering transition in random-field Ising systems has been a puzzle for some time.[1-4] Recent rigorous work[5] has provided convincing justification for an early argument of Imry and Ma that there will be an equilibrium ordered phase for low temperatures and weak random fields in dimensionality d greater than two.[5] The experimental situation[3,4,6] (realized by diluted antiferromagnets in a field)[7] has been clouded by difficulties in reaching equilibrium even at temperatures close to T_{c0}, the transition temperature in zero random field.[3,4,6] Furthermore, recent experiments[6] on $Mn_{0.75}Zn_{0.25}F_2$ have demonstrated that the system falls out of equilibrium very suddenly: Onset of hysteretic behavior on time scales of minutes occurs as the temperature is lowered by about 1% near the ordering temperature, T_c. Although logarithmically slow growth of ordered domains has been predicted to occur if the system is instantaneously quenched to below its ordering temperature,[8,9] there is at present no understanding of what limits the growth of correlations in the critical region.[10]

In this paper I present scaling and renormalization-group arguments based on the assumption of a second-order transition controlled by a zero-temperature fixed point. As a consequence of anomalous growth of the free energy in a correlation volume, I obtain general scaling relations between static quantities and a prediction of dramatic critical slowing down as T_c is approached with the characteristic relaxation time, τ, diverging as

$$\tau \sim \exp\{C/|T-T_c|^{\nu\theta}\}. \tag{1}$$

Here ν is the exponent of the correlation length $\xi \sim |T-T_c|^{-\nu}$ and θ is the "violation of hyperscaling" exponent which controls the anomalous growth of the free energy in a correlation volume leading to

$$2-\alpha = (d-\theta)\nu, \tag{2}$$

with α the specific heat exponent.

The basic observation[1,2] is that at long length scales the static random-field–induced fluctuations dominate over the dynamic thermal fluctuations. As the temperature is increased through T_c, the disordering thus occurs primarily by reversals of the *static* local magnetization (or, generally, the order parameter). The system is considered at a length scale of the correlation length, ξ; the singular part of the effective free energy F_ξ of a correlation volume ξ^d is, schematically, a function of the total magnetization M_ξ in this region. For conventional systems, the characteristic scale of variation of F_ξ is simply set by the thermal fluctuations, i.e., $F_\xi \sim T$, leading to the usual hyperscaling relation $d\nu = 2-\alpha$. However, in the presence of random fields, the scale of variation of F_ξ will be set by the random-field fluctuations which dominate over the thermal fluctuations and cause the scale of F_ξ to grow with ξ as ξ^{θ} leading to the modified hyperscaling relation Eq. (2). The characteristic scale of M_ξ over which F_ξ varies is of order $t^{\beta}\xi^d = \xi^{d-\beta/\nu}$, with β the magnetization exponent. One expects that the static correlation functions of the spins S, will scale as

$$\overline{\langle S(0)\rangle\langle S(x)\rangle} \sim x^{-d+2-\tilde{\eta}}\Gamma_S(x/\xi)$$

(where the bar denotes averaging over the randomness). I have introduced an exponent $\tilde{\eta}$ which plays the analogous role in describing the dominant static fluctuations here to the one that η does for the thermal fluctuations in conventional systems.[11] Since F_ξ will typically be asymmetric, at a scale ξ $\langle S\rangle$ will be of order M_ξ/ξ^d, so that $\tilde{\eta}$ must be related to β by $\beta = \frac{1}{2}(d-2+\tilde{\eta})\nu$.

The thermal fluctuations around the minimum of F_ξ will typically be small because of the large curvature of F_ξ on the scale of T. However in a small fraction of order T/ξ^{θ} of the correlation volumes, there can be two minima of F_ξ which by chance differ in free energy by only of order T. In these active regions equilibrium fluctuations should occur back and forth between the two minima. The contribution of these large rare fluctuations to the averaged thermal correlation function is suppressed by the same factor T/ξ^{θ} which will suppress the magnitude of the common small fluctuations. Since at T_c the correlations of the thermal fluctuations must be independent of ξ, and from the above argument for $x \sim \xi$, the connected correlation function scales as $C_T \sim \Gamma_T(x/\xi)/x^{d-2+\eta}$ with $\eta = \tilde{\eta}$

$+\theta$.

The scaling of the susceptibility, $\chi \sim |T-T_c|^{-\gamma}$, can be obtained either by integrating up C_T or from the scale of the curvature of $F_\xi(M_\xi)$; these simply yield the usual scaling relations $\gamma=(2-\eta)\nu=2-\alpha-2\beta$.

In order to obtain the dynamic scaling behavior, we must consider the characteristic time scales for the fluctuations in equilibrium. For most of the correlation volumes, the thermal fluctuations will primarily occur about the lowest minimum of F_ξ. However, in the rare active regions, equilibrium fluctuations will occur back and forth between two minima of F_ξ since they differ in free energy by only of order T. These fluctuations will be thermally activated over free-energy barriers. It is assumed that the *whole distribution of F_ξ scales as ξ^θ*. Thus the free-energy barriers will have typical height ξ^θ. The dominant slow fluctuations at scale ξ will therefore occur on exponentially long time scales as in Eq. (1). As T_c is approached from above, fluctuations on scales longer than the correlation length can proceed by many approximately independent activations. Since the distribution of these barriers will have a width of order ξ^θ, the dynamic scaling behavior will be on a logarithmic time scale rather than with a single characteristic relaxation time. Thus we expect that for $T\to T_c$ the ac order parameter susceptibility will scale as

$$\chi(\omega,T)\sim\xi^{2-\eta}X(\ln\omega/\xi^\theta), \tag{3}$$

with X a universal scaling function. Various choices for the characteristic relaxation time τ can be made leading to different constants C in Eq. (1). However, since the distribution of barriers is essentially a static quantity there will be a universal amplitude ratio relating each particular $\ln\tau$ to the singular part of the free-energy density, f. Thus $T_c\ln\tau/\xi^d f$ should approach a universal constant as $T\to T_c^+$. At the critical point, since χ must be independent of ξ we have $\chi(\omega)\sim|\ln\omega|^{(2-\eta)/\theta}$.

Although I have argued that there are three independent exponents, there are several inequalities between them. By generalization of the Widom scaling law, we expect the surface tension in the ordered phase to scale as $\xi^{\theta-d+1}$; the requirement that it go to zero as $T\to T_c^-$ implies $\theta\leq d-1$. Another inequality for θ is suggested by consideration of the random-field contribution to the free energy in a correlation volume which should scale as ξ^θ. If the local magnetizations were uncorrelated with the random fields this would scale as $\xi^{-\beta/\nu+d/2}$, since the total random field in the region scales as $\xi^{d/2}$. Since correlations between the magnetization and the random fields should not decrease the contribution to the free energy, we expect that[12] $\theta\geq d/2-\beta/\nu$ which implies $\theta\geq 2-\eta$ and $2\eta\geq 2+\tilde\eta$. This last inequality has been proven for a class of models by Schwartz and Soffer.[13] Together with the requirement that the disconnected correlations fall off at T_c, i.e., $d-2+\tilde\eta\geq 0$, it implies that $\eta\geq(4-d)/2$.

I now turn to a discussion of the critical behavior of the random-field Ising model in the framework of the renormalization group. The main assumption that I have made is that the scale of variation of the effective free energy in a correlation volume scales as ξ^θ with an exponent θ which is independent of $\tilde\eta$ and ν. This is a consequence of the observation that the important competition yielding the phase transition is between the exchange interactions and the random field which implies that the controlling critical fixed point is at zero temperature with the temperature irrelevant[14] with some eigenvalue $\lambda_T<0$. This is known to be the case both in mean-field theory and near six dimensions. In this limit it is just a simple statement of the formal observation that the most divergent terms in a perturbation expansion about mean-field theory are the "tree" graphs, and it is implicit in the $(6-\epsilon)$-dimensional reduction analysis.[2] Thus the transition as a function of random-field strength, h, at $T=0$ and $h=h_{c0}$ should control the behavior of the finite-temperature transition as well and we thus consider approaching the critical point by varying $\delta\equiv h-h_c$.

Scaling laws can be derived by renormalizing until a scale of order of the correlation length at which point the system is outside the critical region and we can match to noncritical quantities. However, since the renormalized temperature will be small, quantities which are singular as $T\to 0$ even outside the critical region will have anomalous scaling laws arising from their singular dependence on the renormalized temperature; i.e., the temperature is dangerously irrelevant.[15] For example, the fact that the free-energy density $f=(T/v)\ln Z$ is nonsingular as $T\to 0$ rather than $\ln Z$ yields the modified hyperscaling relation with the identification $\theta=-\lambda_T$.

The static scaling laws should be valid for $2<d<6$. Although they are not based on the $6-\epsilon$ expansion, they have the same form as the perturbative results in $6-\epsilon$ if we substitute $\theta=-\lambda_T=2$ in this limit which has been shown to be true to all orders in ϵ.[14] However, the modified scaling relations are seen to be simple general consequences of a zero-temperature fixed point and do not depend on dimensional reduction *per se*. In particular, in contrast to various suggestions in the literature,[16] there is no reason to believe that away from six dimensions the random-field exponents $\nu(d)$ and $\eta(d)$ are related to the exponents of the pure system in $d-\theta$ dimensions.

The dynamical critical behavior arises in a similar manner to the statics. We consider the characteristic time $\tau(L,\delta,T)$ for relaxation of fluctuations of scale L with δ small. For $L\sim\xi$, we renormalize to a scale of

order ξ at which point we are outside the critical region. The logarithm of the renormalized relaxation rate will then scale as the inverse of the small renormalized temperature because of noncritical energy barriers in the ground state. Thus we have $\ln\tau(\xi) \sim \xi^{\theta}/T$ so that the exponential dependence on ξ completely swamps the usual power-law renormalization of the time scale.[17,18] For $L \gg \xi$ and $T > T_c$ $\ln\tau(L) \sim \ln\tau(\xi)$, while for $T < T_c$ in this limit the behavior will be more complicated and dominated by domain walls; it will not be discussed in detail here.

In the critical region, $L \ll \xi$, scale invariance of the barriers implies that $\ln\tau(L) \sim L^{\theta}$. Thus at T_c the relaxation between metastable (or similar free energy) configurations in a system of size L^d can involve magnetization changes of order $L^{-\beta/\nu}$ (which decreases slowly with L in $d=3$ since β/ν is expected to be small) and time scales $\tau \sim \exp(L^{\theta})$. At a first-order transition, on the other hand, fluctuations between the phases involve magnetization changes of order 1 and time scales $\tau \sim \exp(L^{d-1})$ (because of creation of an interface of area L^{d-1}). These similarities between the first- and second-order cases and large sample-to-sample variations[19] may give rise to potential difficulties in interpreting Monte Carlo simulations without extensive study of finite-size effects. In particular the recent simulations[19] which were tentatively interpreted in terms of a first-order transition may well be consistent with a second-order transition.

There is a simple extension of our equilibrium results to the nonequilibrium behavior at long times, t, following a quench into the critical region. We are interested in the time-dependent growth of the maximum scale R at which the correlations are approximately in equilibrium. It is natural to assume that on shorter scales there exists local equilibrium. Thus, we expect that for R large but much less than ξ, the time scale t for correlations to reach the scale R will be similar to that in equilibrium, i.e., of order $e^{R^{\theta}}$ so that $R(t) \sim (\ln t)^{1/\theta}$. This is to be contrasted with the previously predicted behavior below T_c,[8,9] where for $R \gg \xi$, the coherence length of the long-range order grows as $R(t) \sim K \ln t$ because of the slow growth of regions of differing magnetizations which are separated by domain walls. By use of the critical forms of the surface tension and the coupling between the magnetization and the random field, it is found that near T_c, $K \sim \xi^{1-\theta}$. The behavior as the system is slowly cooled through T_c is thus rather sensitive to the sign of $\theta - 1$. If $\theta < 1$, then at asymptotically long times the coherence develops most rapidly near T_c, while if $\theta > 1$, the coherence will develop more rapidly as T is decreased.

The behavior discussed above is a consequence of the crossover between the critical fixed point at $h = h_{c0}$ and $T=0$ and the ordered fixed point at $h=0$, $T=0$ near which $K \sim T/h^2$.[8,9] If the random field is small (as is often the case experimentally), then the behavior relatively near to T_c will also be affected by crossover away from the pure (generally zero random field) critical fixed point at $h=0$, $T=T_{c0}$. Our results will then be valid in the critical region if we measure all lengths in units of the crossover length $L_0 \sim h^{2/(2-\eta_0)}$ and times in units of $t_0 \sim L_0^{z_0}$, where η_0 and z_0 are the exponents in zero random field. Physically, L_0 is the length scale at which appreciable metastability starts to occur. The logarithmic growth of coherence in all of the regimes can be simply derived by renormalization to the length scale R and then use of the appropriate behavior at the renormalized temperature and field. The apparent nonequilibrium phase boundary has been observed[20] to scale as the shift in T_c for small fields as expected.

I have argued that the exponential growth of relaxation times with length scale and the consequent logarithmic growth of the coherence length with time occurs in the critical region as well as below T_c. This has several experimental consequences. The first is that on cooling the system will fall out of equilibrium on experimental time scales (say minutes) in a very narrow temperature range. This is consistent with the recent experiments[6] on $Mn_{0.75}Zn_{0.25}F_2$. These experiments do not, therefore, necessarily imply a discontinuous transition as has been suggested by the authors. Another consequence of the slow dynamics in the critical region is that the maximum order-parameter coherence length observed when the system is cooled to $T < T_c$ and held there will in general depend both on the waiting time at the final temperature and on the cooling path (e.g., whether or not h is varied) and cooling rate. For relatively rapid cooling and $\theta > 1$, as seems to be the case experimentally,[6] the final coherence length should be similar for different paths from the disordered side, as observed.[6] With this interpretation, however, the apparent lack[6] of the predicted[8] logarithmic growth with time below T_c cannot easily be explained. Unfortunately, crossover effects from the relatively weak random fields and other corrections to asymptotic behavior may complicate the analysis. Logarithmic time dependence of the capacitance has been observed[21] just below T_c in a related material, $Fe_{0.68}Zn_{0.32}F_2$; however, a comparison of the magnitude of the effect with theory has not been made. Finally, the sudden drop of the Bragg peak intensity observed on warming through T_c after cooling in zero field[6] can be explained as a consequence of the long time scales for critical fluctuations below T_c.

In the future, it should be possible to investigate the dynamics of the random-field magnetic systems both below T_c and in the critical region by NMR or other methods which combined with systematic real-time measurements at long times and inelastic neutron

scattering at short times could span a wide dynamical range and quantitatively test the theoretical predictions. In particular, in diluted antiferromagnets in a field,[3,4,6] the ac magnetic susceptibility should scale as the specific heat: $\chi_F(\omega,T) \sim (T-T_c)^{-\alpha} \sum (\ln\omega/\xi^\theta)$. This implies that the apparent width of the transition due to finite-frequency measurements should scale as $\Delta T \sim |\ln\omega|^{-1/\theta\nu}$.

The ideas presented here should also be applicable to experiments on two-fluid phase separation in porous media or in gels[22] if the dynamics of the gel itself are ignored. A potential advantage of these systems over the magnetic ones is that the exponent β (which is likely to be small) can be measured directly via the shape of the coexistence curve.

After this work was completed, a preprint was received from Villain[23] which makes similar predictions. The assumptions are stated in a rather different way, however, which perhaps makes them seem less natural. Another recent preprint[24] discusses a scaling analysis of the static behavior similar to that given here.

Finally, I note that some of the features of the scaling behavior discussed here may be applicable to transitions controlled by zero-temperature fixed points in other disordered systems, such as the spin-glass to ferromagnetic transition.

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