# Mass Generation by Merons in Quantum Spin Chains and the O(3) $\sigma$ Model

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We argue that the mass gap in the O(3) $\sigma$ model is produced by condensation of vortices of topological charge $\frac{1}{2}$ (merons). This mechanism is not effective at topological angle $\theta=\pi$, supporting arguments that this model is massless. These two models are the large-spin limit of integer (half-integer) quantum spin chains.

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The two-dimensional O(3) nonlinear $\sigma$ model has long been of interest to particle theorists because of its strong analogy with four-dimensional Yang-Mills theory (including dynamical mass generation and topological effects).[1,2] It is of interest to condensed-matter theorists, not only as a model of a classical two-dimensional ferromagnet,[3] but also as the large-$s$ limit of a spin-$s$ antiferromagnetic quantum chain[4]:

$$H=\sum_n \mathbf{S}_n\cdot\mathbf{S}_{n+1},\quad \mathbf{S}_n^2=s(s+1). \tag{1}$$

A simple proof[5] of this correspondence uses the fact that at large $s$, neighboring spins are almost antiparallel to define

$$\boldsymbol{\phi}=(\mathbf{S}_{n+1}-\mathbf{S}_n)/2s,\quad \mathbf{l}=(\mathbf{S}_{n+1}+\mathbf{S}_n)/2\Delta \tag{2}$$

($\Delta$ is the lattice spacing). In the continuum limit, these obey the commutation relations of the scalar fields ($\boldsymbol{\phi}^2=1$) and rotation generators of the O(3) $\sigma$ model. In this limit the Hamiltonian becomes $2\Delta s H_\sigma$ where $H_\sigma$ is the $\sigma$-model Hamiltonian:

$$H_\sigma=\int dx\left\{\frac{g^2}{2}\left[\mathbf{l}-\frac{\theta}{4\pi}\frac{\partial\boldsymbol{\phi}}{\partial x}\right]^2+\frac{1}{2g^2}\left(\frac{\partial\boldsymbol{\phi}}{\partial x}\right)^2\right\} \tag{3}$$

with coupling $g^2=2/s$ and topological angle $\theta=2\pi s$ (effectively 0 or $\pi$ for $s$ integer or half integer). A more realistic spin Hamiltonian includes anisotropy:

$$H\rightarrow H+\sum_n[a(S_n^z)^2+bS_n^zS_{n+1}^z]. \tag{4}$$

Again, taking the large-spin continuum limit gives

$$H_\sigma\rightarrow H_\sigma+(m^2/2g^2)\int dx(\phi_z)^2, \tag{5}$$

with $m^2=2(a+b)/\Delta^2$.

The corresponding Euclidean-space theory with Lagrangean

$$\mathscr{L}=\frac{1}{2g^2}[(\partial_\mu\boldsymbol{\phi})^2+m^2\phi_z]+\frac{i\theta}{4\pi}\epsilon^{\mu\nu}\boldsymbol{\phi}\cdot(\partial_\mu\boldsymbol{\phi}\times\partial_\nu\boldsymbol{\phi}) \tag{6}$$

will be the subject of this paper. Instanton effects have been widely discussed in the isotropic model ($m^2=0$).[2] However, the same infrared problems that plague four-dimensional Yang-Mills theory prevent instanton calculations from being reliable (in particular the integral over instanton size is infrared divergent). The instanton, of topological charge $q=1$

$$[q=(1/4\pi)\int d^2x\,\epsilon^{\mu\nu}\boldsymbol{\phi}\cdot(\partial_\mu\boldsymbol{\phi}\times\partial_\nu\boldsymbol{\phi})], \tag{7}$$

can be thought of as being made up of a meron-antimeron pair, each with $q=\frac{1}{2}$.[2] However, isolated merons are not classical solutions because of ultraviolet singularities. Merons which satisfy the classical equations outside their cores can be constructed: They have long-range Coulomb interactions. They are not generally considered responsible for mass generation in the $\sigma$ model, despite some analogy with the Coulomb gas. As we shall see, the situation is much more tractable with the anisotropic term present. Merons in the anisotropic model have been discussed elsewhere.[6] In this paper we try to present convincing arguments that, despite the usual infrared problems, they *are* responsible for mass generation. Furthermore we show that this mechanism is ineffective at $\theta=\pi$, supporting arguments that this model is massless. This latter fact is crucial to an understanding of antiferromagnets[4] and also of the existence of extended states in the quantum Hall effect.[7]

Let us consider first the case $\theta=0$. There are now two mass scales: $m$ and the dynamical mass-generation scale $\mu^2=\Lambda^2\exp(-2\pi/g^2)$ ($\Lambda$ is the ultraviolet cutoff). In either limit $m^2>>\mu^2$ or $m^2<<-\mu^2$, $g_{\rm eff}(|m^2|)$ becomes weak and perturbation theory is reliable. For $m^2$ large and negative we find $\langle\phi_z\rangle=\pm1$ (the discrete symmetry $\phi_z\rightarrow-\phi_z$ is spontaneously broken). The tree-level spectrum consists of two particles $(\phi_x,\phi_y)$ of mass $m$. In the other limit of $m^2$ large and positive it is convenient to write

$$\boldsymbol{\phi}=((1-\phi_z^2)^{1/2}\cos\alpha,(1-\phi_z^2)^{1/2}\sin\alpha,\phi_z). \tag{8}$$

$\phi_z$ has the mass $m$; the angle $\alpha$ is a massless field which apparently has a vacuum expectation value, breaking the U(1) symmetry. While quantum effects must restore the symmetry (Coleman's theorem[8]) a massless decoupled boson should be present[9] at sufficiently weak coupling [i.e., large $m^2/\mu^2$]. Since the isotropic model is massive,[10] a phase transition must occur at some positive value of $m^2$ [of $O(\mu^2)$]. [Similarly, a second phase transition must occur at a negative value of $m^2$, of $O(\mu^2)$, where the discrete symmetry $\phi_z \to -\phi_z$ is spontaneously broken.] We wish to argue that the first transition occurs as a result of condensations of merons.

To do this it is convenient to regard the Euclidean field theory as the continuum limit of a *classical* two-dimensional lattice ferromagnet with action[11]

$$S = T^{-1}\sum_{\mathbf{x},\mu}[\mathbf{S}(\mathbf{x})\cdot\mathbf{S}(\mathbf{x}+\Delta\mu)+\Delta^2 m^2 S_z(\mathbf{x})^2]. \quad (9)$$

We now have two free parameters: $T$ and $\Delta^2 m^2$. The expected phase diagram[11] is shown in Fig. 1. At $\Delta^2 m^2 \to \infty$, $S_z \to 0$ and we obtain the $xy$ model. This is well known to have a massless phase for $T < T_{xy}$.[12] This critical line joins smoothly with the field-theory result as $T \to 0$: $m_c^2(T) \sim (1/\Delta^2)e^{-2\pi/T}$. (Similarly at $m^2\Delta^2 \to -\infty$, $S_x, S_y \to 0$ and we obtain the Ising model. This second critical line also approaches $T=0$ exponentially.) The nature of the transition in the $xy$ model is well understood. It is induced by vortex condensation. A vortex is a planar spin configuration which, at infinity, has spins aligned with the azimuthal direction, $\theta$ (Fig. 2). These configurations have an infrared-divergent action. A vortex-antivortex pair separated by a large distance $r$ ( $>>\Delta$) has action[12]

$$S = \text{const}/T + (\pi/2T)\ln(r/\Delta). \quad (10)$$

The combined spin-wave + vortex partition function is conveniently represented (in a large-distance approximation) by the sine-Gordon model:

$$S = \int d^2x\,[\tfrac{1}{2}(\partial_\mu\phi)^2 + \gamma\cos\beta\phi], \quad (11)$$

with $\beta^2 = 2\pi^2/T$, $\gamma \to \infty$. Expanding the partition function in powers of $\gamma$ gives back the Coulomb gas, $\gamma$ playing the role of fugacity. A renormalization-group analysis in the $\gamma,\beta$ plane yields exact critical exponents[12]:

$$\langle e^{-i\rho(x)}e^{i\rho(y)}\rangle \sim (x-y)^{-4} \quad (\text{at } T=T_c), \quad (12)$$

where $\rho(x)$ is the charge density. At $T \gtrsim T_c$ there is a mass gap[12]

$$m \sim \exp[-\text{const}/(T-T_c)^{1/2}] \quad [\nu=\infty]. \quad (13)$$

Vortices are expected to disappear in the continuum limit of the *planar* model since the action [Eq. (10)] blows up as $\Delta \to 0$. However, we expect the critical line in the $m^2, T$ plane to continue down to the origin. If it did not, some kind of multicritical end point would have to appear at finite $T$. The most conservative assumption[11] is that the whole line is in the same universality class, including the continuum limit. This seems to raise a paradox since planar vortices do not contribute in the continuum limit and yet some sort of vortices are necessary to produce the phase transition.

The resolution is given by nonplanar vortices[13] that have finite action (for vortex-antivortex pairs) in the continuum limit. By lifting off the plane, an ultraviolet singularity is avoided. In general, as $T$ is decreased along the critical line we expect the most thermodynamically favored vortex to become increasingly nonplanar. A vortex configuration obeying the continuum equations of motion exists. At infinity the spins lie along the equator ($\phi_z = 0$); at the origin an ultraviolet singularity is avoided by the spins going to the north or south pole ($\phi_x = \phi_y = 0$). If we introduce spherical coordinates,

$$\boldsymbol{\phi} = (\sin\beta\cos\alpha, \sin\beta\sin\alpha, \cos\beta), \quad (14)$$

the Lagrangean becomes

$$\mathscr{L} = (1/2g^2)[(\nabla\alpha)^2 + \sin^2\alpha(\nabla\beta)^2] + (m^2/2g^2)\cos^2\alpha. \quad (15)$$

At infinity $\alpha = \pi/2$ and $\beta = \phi$ (the spatial angle). If we make a circular-symmetry *Ansatz* $\beta = \phi$, $\alpha = \alpha(r)$,

$$\mathscr{L} = \left(\frac{1}{2g^2}\right)\left[\left(\frac{d\alpha}{d}r\right)^2 + \frac{\sin^2\alpha}{r^2} + m^2\cos^2\alpha\right]. \quad (16)$$

It is elementary to prove that a solution exists with $\alpha(r)$ increasing monotonically from 0 at $r=0$ to $\pi/2$

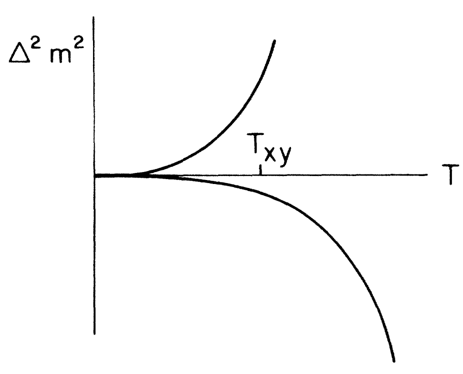


FIG. 1. Phase diagram of the anisotropic lattice spin system.

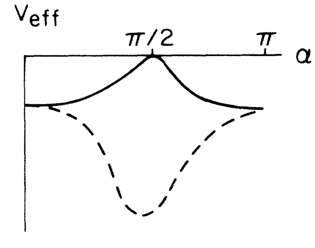


FIG. 2. Effective potential $V(\alpha,r)$ for the classical particle problem. Solid line is at $r \to \infty$; dotted line at $rm << 1$.

at $r=\infty$. The classical equation is that of a particle of position $\alpha$ at time $r$ moving in a time-dependent potential

$$V_{\text{eff}}(\alpha,r)=-m^2\cos^2\alpha-(1/r^2)\sin^2\alpha$$

(Fig. 2) with a time-dependent frictional force $-r^{-1}d\alpha/dr$. At $r \ll m^{-1}$ the mass term is irrelevant and $\alpha(r)\to r/\rho$, the form of an instanton of size $\rho$ in the isotropic theory. At $r\to\infty$, $V_{\text{eff}}\to -m^2\cos^2\alpha$ so that the particle is driven to one of the valleys at $\alpha=0,\pi$ unless $\rho$ is chosen to have a special value [of $O(m^{-1})$] so that the particle ends up on top of the hill ($\alpha=\pi/2$). To prove this we see that for $\rho \gg m^{-1}$ the particle is always close to $\alpha=0$ since this is true trivially for $\rho=\infty$. On the other hand for $\rho \ll m^{-1}$ the particle gets to $\alpha=\pi$ before the mass term becomes effective since this is the behavior of the instanton solution in the isotropic theory. By continuity there must be some value of $\rho$ such that the particle just makes it to the hilltop as $r\to\infty$. By dimensional analysis the action of a vortex-antivortex pair is

$$S=(1/g^2)[\text{const}+(\pi/2)\ln rm] \quad (r \gg m^{-1}). \tag{17}$$

In spherical coordinates the topological charge becomes

$$q=(1/4\pi)\int d^2\epsilon_{\mu\nu}\,\partial_\mu[\cos\alpha\,\partial_\nu\beta]. \tag{18}$$

This can clearly be written as a line integral. The instanton of the isotropic theory gets equal contributions to $q$ from the line integral at $r=0$ and that at $r=\infty$. However, the vortex only gets a contribution to $q$ from $r=0$. Thus $q=\pm\frac{1}{2}$ for a vortex; it is a "meron." There are clearly two types of meron for each vortex sign (four objects in total) since the spins may move to the north or south pole at $r=0$ and may circulate in either direction around the equator at $r=\infty$.

In the weak-coupling regime, the one-loop correction to the instanton action should replace $g^2$ by $g^2_{\text{eff}}(m)\ll 1$. Thus the meron gas is very dilute ($\gamma \ll 1$) and has a low temperature; the theory is in the massless boson phase. All higher loop corrections to the instanton should also be infrared finite and small. As $m^2$ is reduced the effective vortex temperature $T\sim g^2_{\text{eff}}(m)$ is increased until finally the vortices condense. Note that this is more than a mere hand-waving description; the above arguments determine exact critical exponents [Eqs. (12) and (13)]. Thus we may say that the mass gap in this anisotropic $\sigma$ model near the critical point is caused by meron condensation. (As $m^2$ is reduced to zero this explanation of the mass gap becomes less meaningful.)

The presence of two types of vortex ($q=\pm\frac{1}{2}$) for each vortex sign does not affect anything. In general a Coulomb gas of two species with different charges and chemical potentials is equivalent to a sine-Gordon model with

$$\mathscr{L}=\tfrac{1}{2}(\partial_\mu\phi)^2+\gamma_1\cos\beta_1\phi+\gamma_2\cos\beta_2\phi. \tag{19}$$

In this case $\gamma_1=\gamma_2$, $\beta_1=\beta_2$ so that $\gamma$ is simply doubled.

Let us now consider a nonzero topological angle, $\theta$. In this case the $q=\pm\frac{1}{2}$ vortices have $\gamma_\pm=\gamma\times\exp(\pm i\theta/2)$, so that altogether the net chemical potential is $\gamma_{\text{net}}=2\gamma\cos\theta/2$. Increasing $\theta$ reduces the $\gamma_{\text{eff}}$ and thus tends to increase $T_c$. Finally at $\theta=\pi$, $\gamma_{\text{eff}}=0$ and (naively) the theory is always in the massless phase. Note that meron and antimeron precisely cancel. This cancellation continues to hold when perturbative corrections to the meron are considered. In fact, any smooth configuration of vortex charge 1 has half-integer topological charge (since a hemisphere can only be mapped onto a sphere with half-integer winding number). Thus a precise cancellation occurs between positive and negative topological charge configurations. However, there *are* double vortices with $q=0,\pm1$. Their contributions do *not* cancel and so they may produce the phase transition. But doubling the vortex charge raises the critical temperature (by a factor of 4 if $\gamma=0$). Thus we *may* reasonably conclude from this argument that $m_c^2(\theta)$ is a decreasing function (for $0\le\theta\le\pi$).[14] In fact, arguments based on the study of quantum spin chains[4] suggest that $m_c^2(\pi)=0$ and that the *xy* and Ising critical points merge at $\theta\to\pi$. A critical theory near $\theta=\pi$ was deduced elsewhere.[15]

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