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Resonances of Chaotic Dynamical Systems

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We present analytic properties of the power spectrum for a class of chaotic dynamical systems (Axiom-A systems). The power spectrum is meromorphic in a strip; the position of the poles (resonances) depends on the system considered, but only their residues depend on the observable monitored. In relation with these results we also discuss the exponential or nonexponential decay of correlation functions at infinity. In conclusion, it appears desirable to analyze the decay of correlation functions and the possible analyticity of power spectra for physical time evolutions, and for computer-generated simple dynamical systems (non-Axiom-A in general).

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A differentiable dynamical system is a time evolution $x(t)=f^t x(0)$, where t may be an integer (discrete-time system, f^t is the tth iterate of a differentiable map f) or a real number [continuous-time system, $x(t)$ usually defined by a differential equation $dx/dt=F(x)$]. The system lives on a manifold M, which is often Euclidean space, or may be infinite dimensional.[1] If B,C are differentiable functions on M (interpreted as *observables*) we introduce the *correlation function*

$$\rho_{BC}(t)=\langle B(f^{t+\tau}x)C(f^{\tau}x)\rangle_\tau-\langle B\rangle\langle C\rangle.$$

It is assumed that the time averages $\langle X\rangle_\tau=\lim T^{-1}\int_0^T d\tau\, X$ converge, defining a probability measure ρ, invariant under time evolution, and ergodic. We may then write $\langle B\rangle=\int\rho(dx)B(x)=\rho(B)$, and

$$\rho_{BC}(t)=\rho((B\circ f^t)\cdot C)-\rho(B)\rho(C).$$

Let also

$$\hat\rho_{BC}(\omega)=\int dt\, e^{i\omega t}\rho_{BC}(t),$$

where we replace the integral by a sum for discrete-time systems. Notice that $\hat\rho_{BB}$ is the *power spectrum* of the signal $B(x(t))$.

What are the decay properties of ρ_{BC}? (Exponential?) What are the analyticity properties of $\hat\rho_{BC}$? These are very hard mathematical questions in general. In this Letter, some new results are reported, coming from the work of Pollicott[2] and the author,[3] and various comments and questions are proposed. We shall be concerned with an important class of chaotic dynamical systems called hyperbolic, or Axiom-A, systems.[4] A famous example is the *geodesic flow on a manifold of negative curvature,* i.e., the dynamics of the frictionless motion of a particle moving with unit speed on a compact manifold with everywhere negative sectional curvature. For Axiom-A systems there is a natural choice of ρ (stable under small stochastic perturbations) which belongs to the class of *Gibbs states.*

Theorem.—Let (f^t) be an Axiom-A time evolution on M, and ρ a Gibbs state. Then (a) there is $\delta>0$ such that $\hat\rho_{BC}(\omega)$ is meromorphic (i.e., holomorphic except for poles) in the strip $|\mathrm{Im}\,\omega|<\delta$. The position of the poles is independent of BC. If the system is mixing, there are no poles on the real axis. (b) If ν is a simple pole of $\hat\rho_{BC}$, its residue is (usually) of the form $\sigma_-(B)\sigma_+(C)$, where σ_- and σ_+ are (Schwartz) distributions on M behaving covariantly under time evolution,

$$f^t\sigma_-=e^{-i\nu t}\sigma_-,\quad f^t\sigma_+=e^{i\nu t}\sigma_+.$$

$\sigma_\pm$ can be characterized by an extension of the Gibbs property of statistical mechanics.

In (a), mixing means that $\rho_{UV}(t)\to 0$ when $|t|$

$\to \infty$ for all choices of the functions U, V. In (b), "usually" means that for continuous-time systems we have omitted a condition which is generally (perhaps always) true. If the support of ρ is an attractor (or more generally a *basic set*[4]) then the distributions σ_- and σ_+ also have this attractor (or basic set) as support.

The proof of (a) is in large part due to Pollicott[2] and the rest to the author.[3] These proofs use the relation between the dynamics of Axiom-A systems and the equilibrium statistical mechanics of certain one-dimensional spin systems (this explains why we speak of Gibbs states in both cases). Specifically, the *time* correlation function for an Axiom-A system is identical with the *space* correlation function for a one-dimensional lattice spin system with exponentially decreasing interactions. (This is true at least for discrete-time dynamical systems; for continuous-time systems the situation is slightly more complicated. See Bowen,[5] Ruelle,[6] and references given there.) If we express ρ_{BC} in terms of the *transfer matrix* L for the spin system, the poles of $\hat{\rho}_{BC}$ are found to be simply related to the eigenvalues of L. (Since the interactions do not have finite range, L is an operator in an infinite-dimensional Banach space). Incidentally, there is a kind of Fredholm determinant associated with L; its inverse is called a *zeta function* and the poles of this zeta function are simply related to those of $\hat{\rho}_{BC}$.

The complex poles of $\hat{\rho}_{BC}$ are naturally interpreted as *resonances*. For discrete-time dynamical systems $\hat{\rho}_{BC}$ is periodic, and (a) implies that the poles stay a finite distance away from the real axis. This corresponds to the known fact that ρ_{BC} is exponentially decreasing at infinity.[6] For continuous-time systems, the poles may come arbitrarily close to the real axis, preventing $\rho_{BC}(t)$ from decaying exponentially at infinity.[7] On the other hand, the geodesic flow on a manifold with *constant* negative curvature has $\hat{\rho}_{BC}(\omega)$ analytic in a strip $|\mathrm{Im}\omega| < \delta'$.[8] Nothing is known about nonconstant curvature. For a nonmixing system poles are regularly spaced on the real axis; suppose that the system is perturbed to make it mixing, do the poles move to produce a strip of analyticity? Again nothing is known. The one example which we know[7] where the poles of $\hat{\rho}_{BC}$ come arbitrarily close to the real axis does not correspond to an attractor (but to a nonattracting basic set). It is thus still conceivable that the physical correlation functions (for Axiom-A attractors) decay exponentially. If not, what are the kinds of nonexponential decays which really occur? In particular, what physical meaning can one attach to those nonexponential decays?

All these questions appear rather formidable mathematically, in particular in view of their relation with zeta functions. Indeed the zeros and poles of zeta functions are notoriously hard to locate (even though the functions which occur here are closer to the "easy" Selberg zeta function than to the Riemann zeta function).

These considerations suggest trying to investigate numerically the problems raised above. For Axiom-A systems, an indirect attack would be more efficient, and will be discussed elsewhere. The case of simple non-Axiom-A systems, however, invites an immediate analysis. Specifically, one would like to compute the correlation functions $\rho_{x_i x_j}(t)$ for the Hénon map, the Lorentz system, or the simple quadratic map $x \to ax(1-x)$ of the interval [0,1], and the analyticity properties of the Fourier transforms $\hat{\rho}_{x_i x_j}(\omega)$. (Here the observables x_i are just the coordinates of the vector x). A similar study for an experimental signal $u(t)$ would be very interesting, but harder because of the difficulty of obtaining u with sufficient precision (see below). A casual observation of experimental power spectra (from hydrodynamics) sometimes shows regularly spaced bumps or resonances which are very suggestive of complex poles near the real axis. Hopefully, these poles could be located and analyzed precisely.

Let us try to assess the effect of noise (or round-off errors in computer studies) on the correlation function

$$\rho_{BC} \approx T^{-1}\int_0^T B(f^{t+\tau}x)C(f^{\tau}x)d\tau - \langle B\rangle\langle C\rangle.$$

An imprecision ν on x will grow exponentially with time like $\nu e^{\lambda_1 t}$, where λ_1 is the largest Lyapunov exponent (see Ref. 1). A precise determination of $\rho_{BC}(t)$ requires thus

$$t << |\log\nu|/\lambda_1 << T.$$

The second inequality allows the noise to play its role in selecting the probability measure ρ, if ρ is determined by its stability under small stochastic perturbations; see Ref. 1. When the first inequality is violated, one can argue that the noise produces a decay of correlations like

$$\exp[-t\textstyle\sum(\text{positive Lyapunov exponents})].$$

This exponential decay due to the noise combines with the natural decay of $\rho_{BC}(t)$. A numerical study of $\log|\rho_{BC}(t)|$ vs t should thus show two different rates of decay depending on whether t is small or large compared with the characteristic time $|\log\nu|/\lambda_1$.

To conclude, let us mention that Frisch and Morf[9] have also discussed complex poles in the context of dynamical systems, but their poles are for the signal $u(t)$ rather than its Fourier transform (and the approach is completely different).

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