Complex Langevin Solution of the Schwinger Model

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A complex Langevin method proposed previously is applied to the massive lattice Schwinger model. The coupled system is consistently treated by the complex Langevin equations.

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Standard Monte Carlo (MC) sampling appears to be a successful method for studying lattice field theories without fermions. As a result of the properties of a Grassmann algebra (in Euclidean formalism the fermions are represented by the generators of a Grassmann algebra[1]) no direct numerical simulation is feasible. So far, different methods have been proposed to elude the problem.[2] Starting with the Hamiltonian formalism of lattice field theories, Hirsch *et al.*[3] developed a method to treat fermionic systems by direct numerical simulation.

In previous work[4,5] we presented a technique to construct complex fermionic path integrals. The method is based on the Jordan-Wigner representation[6] of fermion operators. Spin-coherent states[7] are chosen to be the intermediate states passing from the Hamiltonian formalism to the complex path integral. Because of a lack of positivity of $\exp(-S)$, MC sampling has to be excluded during investigation of a complex action. Under some conditions on the action, complex path integrals can be treated by solving an equivalent stochastic process.[4,8] Unfortunately there appear factors $1/\sin\theta$ in the Langevin equations associated with the complex path integral proposed in Refs. 4 and 5, which make the numerical treatment difficult.

With the introduction of Cartesian variables (x,y,z) this problem does not appear.[5,9] A short sketch of the derivation of the Langevin equations in a Cartesian representation is published elsewhere.[5] For details we refer to Ref. 9.

It had been shown[5,9] that the complex Langevin method has sufficiently good convergence properties to describe simple fermionic systems: the free relativistic fermion system $(d=2)$ and the Thirring model as an example of a self-interacting system. Here it is our intention to show that the complex Langevin method works well even for gauge-interacting fermions. We chose the Schwinger model for the following reasons. The Schwinger model is the simplest model which shows QCD-like properties. The exactly solvable massless continuum Schwinger model[10] and even the massive one[11] have been extensively studied.[10–12] The lattice Schwinger model itself is often used to test fermion algorithms and to test scaling-limit properties of lattice field theories.[2,13]

To obtain a more simple form for the fermion action we choose the temporal gauge. According to Susskind and Kogut,[14] the fermionic system in an external field will be described by the following Hamiltonian:

$$H=\sum_{n=1}^{N}\left\{-\frac{1}{2a}[c_n^\dagger c_{n+1}U_n(t)+c_{n+1}^\dagger c_n U_n^*(t)]+mc_n^\dagger c_n(-1)^n\right\}. \tag{1}$$

We have chosen the arbitrary phase factor $s_n=(-i)^n$. This implies antiperiodic boundary conditions in Eq. (1) for $N=(k-1)4+2$, k an integer. Following Refs. 4 and 5 we end up with the following fermionic action:

$$S_F=\sum_{n,m=1}^{N,M}\{-\ln\langle x_n^{m+1},y_n^{m+1},z_n^{m+1}|x_n^m,y_n^m,z_n^m\rangle$$
$$-\tfrac{9}{4}[(x_n^m x_{n+1}^m+y_n^m y_{n+1}^m)\mathrm{Re}(U_n^m)+(x_n^m y_{n+1}^m-y_n^m x_{n+1}^m)\mathrm{Im}(U_n^m)]+\overline{m}\times\tfrac{3}{2}(-1)^n z_n^m\}, \tag{2}$$

with $\overline{m}=ma$.

The interaction in the time direction is represented by the logarithm of the coherent-state scalar product, which is a complex number.[4,5,9] To describe the gauge-field dynamics and to quantize it we use the Wilson action.[15] We choose a Cartesian representation $[\hat{x}=\mathrm{Re}(U),\ \hat{y}=\mathrm{Im}(U)]$ for the gauge field for technical reasons, which we will discuss later. The full action in the temporal gauge now reads

$$S=\sum_{n,m=1}^{N,M}\{-\ln\langle x_n^{m+1},y_n^{m+1},z_n^{m+1}|x_n^m,y_n^m,z_n^m\rangle - \tfrac{9}{4}[(x_n^m x_{n+1}^m+y_n^m y_{n+1}^m)\hat{x}_n^m+(x_n^m y_{n+1}^m-y_n^m y_{n+1}^m)\hat{y}_n^m]+\bar{m}\times\tfrac{3}{2}(-1)^n z_n^m\} - \sum_{n,m=1}^{N,M}\beta(\hat{x}_n^m\hat{x}_n^{m+1}+\hat{y}_n^m\hat{y}_n^{m+1}). \quad (3)$$

To be consistent, our intention is to use stochastic quantization for the gauge field, too. Besides, it is not possible to use MC sampling since $\Delta S/\Delta U \in C$. It is very important to realize that not only the fermionic part of the system is described by a complex stochastic process, but even the evolution of the gauge field in the auxiliary time. This is the reason for the use of Cartesian variables to represent the gauge field, since they are much easier to handle in the numerical procedure. For the derivation of the Langevin equations for the gauge system we refer to Ref. 9.

We have been conscious of the fact that to use the temporal gauge on a finite lattice means to make a slight error. Since the pure U(1) lattice theory is an exactly solvable model[16] we are able to obtain an idea about the effects of finite size superposed by the effects of the temporal gauge for the pure U(1) theory. Using an improved Runge-Kutta algorithm[9,17] for multiplicative vector Itô stochastic differential equations we simulated the pure U(1) theory on different lattice sizes. The difference $\langle \mathrm{Re}(U_\Box)\rangle_{N\times N} - \langle \mathrm{Re}(U_\Box)\rangle_\infty$, where $U_\Box$ stands for a product of transformations on a given plaquette, is very small (see, e.g., Fig. 1). The fictitious-time step size h is $h=0.01$. The value $h\approx 0.01$ appears to be the optimal choice, also obtained in earlier calculations,[5,9] since no significant change in the results is obtained by going to $h<0.01$. To obtain the results in Fig. 1 we generated for each point a total of 5000 configurations, discarded the first 1000, and measured every fifth one thereafter.

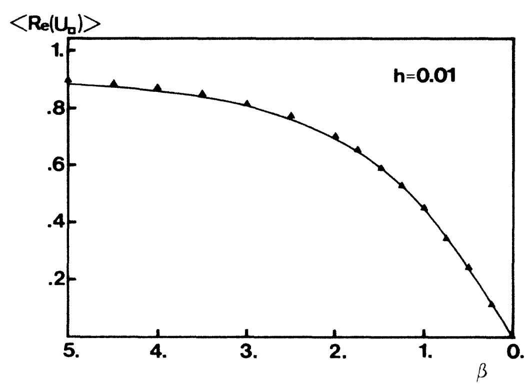

FIG. 1. The value of $\langle \mathrm{Re}(U_\Box)\rangle$ on a 10×10 lattice for the pure U(1) theory (statistical errors are within the symbols); the solid line gives the exact solution for $\langle \mathrm{Re}(U_\Box)\rangle$ for an infinite lattice.

It is one of the important properties of the massless continuum Schwinger model that $\langle \psi(x)\psi^\dagger(x)\rangle \neq 0$.[10,12] From the result for $\langle \psi(x)\bar{\psi}(x)\psi(0)\bar{\psi}(0)\rangle$ given by Baaquie[12] one expects

$$\langle \psi(x)\bar{\psi}(x)\rangle|_{m=0}=\frac{g}{2\pi^{3/2}}e^{\gamma}, \quad (4)$$

where γ is the Euler constant. The order parameter of our system is defined by

$$\langle \psi_n\bar{\psi}_n\rangle = \left\langle \frac{2}{N}\sum_{n=1}^{N} c_n c_n^\dagger(-1)^n\right\rangle, \quad (5)$$

and divided by $\bar{m}$ this operator should be related to (4) in the continuum limit; i.e.,

$$\lim_{\bar{m}\to 0}\frac{\langle \psi_n\bar{\psi}_n\rangle}{\bar{m}}=\frac{1}{2\pi(\pi\beta)^{1/2}}e^{\gamma}. \quad (6)$$

To determine $\langle \psi_n\bar{\psi}_n\rangle/\bar{m}$ with any accuracy would require extensive computation, but it is possible to get significant evidence of whether or not the method is working for the Schwinger model by an investigation of $\langle \psi_n\bar{\psi}_n\rangle$ alone.

Equation (6) implies

$$\langle \psi_n\bar{\psi}_n\rangle = O(\bar{m}), \quad \bar{m}\to 0, \quad (7)$$

Furthermore, we know from previous calculations[13] which behavior for $\langle \psi_n\bar{\psi}_n\rangle$ we should expect, i.e., a splitting depending on β and $\langle \psi_n\bar{\psi}_n\rangle \to 0$ for $\bar{m}\to 0$ for all β; thus the splitting has to vanish for $\bar{m}\to 0$.

Again using the improved Runge-Kutta algorithm we made our calculations on a 10×10 lattice. We generated up to 1000 configurations to thermalize the system. Thereafter we made 10 000 sweeps through the lattice for the measurement of the order parameter, where very fifth configuration was measured. We used a fictitious-time step size of $h=0.01$, which again appears to be favorable. (It should be mentioned that this step size might be much too large if more naive numerical approximations are used.) In Fig. 2 $\langle \psi_n\bar{\psi}_n\rangle$ is plotted for different values of β. We obtain a distinct splitting for $3\geq\bar{m}>0$, which vanishes for $\bar{m}\to 0$. For $\bar{m}>3$ the splitting is not observable

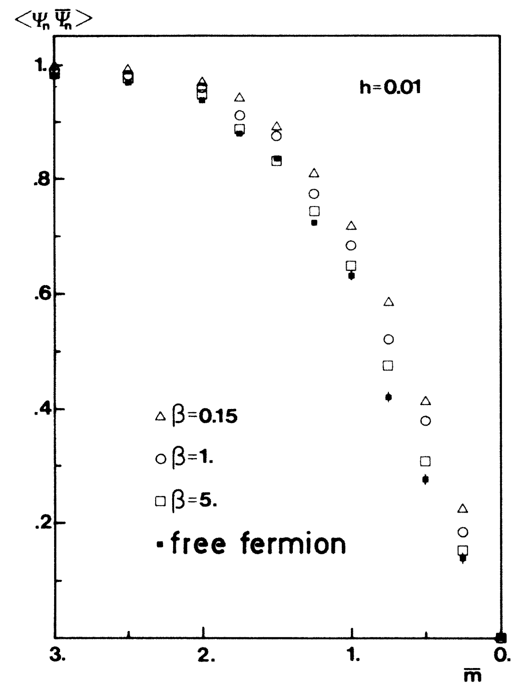

FIG. 2. The order parameter $\langle \psi_n \bar{\psi}_n \rangle$ on a 10×10 lattice for different values of β and for the free theory (statistical errors are within the symbols or indicated by vertical bars).

within our accuracy.

As mentioned above, the evolution of the coupled fermion-gauge system in the fictitious time is described by a complex stochastic process, and thus one might expect large fluctuations, especially in the complex plane. No such effects were obtained. In the coupled system the statistical errors are of about the same magnitude as for the free fermionic system ($d=2$). However, if we should remark that, in general, it seems to be preferable to average over an ensemble of shorter runs and not to make one long run in order to improve the results.

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