Weak Mixing Angles from Semileptonic Decays in the Quark Model

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We use the constituent-quark model to predict the electron spectra in semileptonic meson decays. Particular attention is paid to the end-point region of the spectrum in B decays, which is important to the determination of the $b \to u$ weak mixing angle.

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In the standard model, the four weak mixing angles of the Kobayashi-Maskawa (KM) matrix[1] are, like the quark and lepton masses, fundamental parameters that must be determined by experiment. One of these angles is essentially the Cabibbo angle. Two others can, in principle, be determined from B-meson semileptonic decay since

$$d\Gamma(B^{0,-} \to X^{+,0}e^-\bar{\nu}_e) = |V_{cb}|^2 d\hat{\Gamma}(B^{0,-} \to X_c^{+,0}e^-\bar{\nu}_e) + |V_{ub}|^2 d\hat{\Gamma}(B^{0,-} \to X_u^{+,0}e^-\bar{\nu}_e). \quad (1)$$

Here $d\Gamma$ is the differential decay rate, V_{cb} and V_{ub} are the $b \to c$ and $b \to u$ elements of the KM matrix, and $d\hat{\Gamma}(B^{0,-} \to X_q^{+,0}e^-\bar{\nu}_e)$ is the differential B decay rate induced by a full-strength $b \to q$ current. In the free-quark approximation

$$\hat{\Gamma}(B^{0,-} \to X_q^{+,0}e^-\bar{\nu}_e) \simeq \hat{\Gamma}_{\text{free}}(b \to qe^-\bar{\nu}_e) = (G_F^2 m_b^5/192\pi^3) f(m_q/m_b), \quad (2)$$

where $f(x) = 1 - 8x + 8x^6 - x^8 - 24x^4 \ln x$.

It is known that B decays are dominated by the $b \to c$ transition, so that V_{cb} can be estimated via (1) and (2) from the measured B lifetime. Recent attempts to extract V_{ub} have concentrated on examining the electron spectrum $d\Gamma/dE_e$ in B semileptonic decay, since for $E_e \gtrsim (M_B^2 - M_D^2)/2M_B$ the second term in (1) will dominate. This end-point region of the electron spectrum is, of course, controlled by transitions to a few low-lying confined states of the $q\bar{d}$ (or $q\bar{u}$) system. Thus while free-quark decay should be a good guide to the total rate (2), one would not expect a satisfactory treatment of the end-point region by such methods.

What is needed in this region is a method for explicitly summing over low-mass states X_c and X_u to predict the shapes and strengths of both spectra in the end-point region. In the absence of rigorous methods it is natural to apply the quark model to this task. The constituent-quark model is a model for QCD in the confinement regime which has had considerable success in describing hadronic structure. It is especially well suited (and well tested) for describing the low-mass hadrons like those which control the $B \to Xe\bar{\nu}_e$ end-point spectrum.

The transition matrix element for the process $B^{0,-} \to X_q^{+,0}e^-\bar{\nu}_e$ is

$$T = (G_F/\sqrt{2}) V_{qb} \bar{u}_e \gamma_\mu (1-\gamma_5) v_{\nu_e} \langle X_q(p_X s_X)| j_{qb}^\mu | B(p_B)\rangle, \quad (3)$$

where j_{qb}^μ is the $b \to q$ hadronic current to which we referred above. The hadronic tensor

$$h_{qb}^{\mu\nu} = \textstyle\sum_{s_X} \langle B(p_B)| j_{qb}^{\nu\dagger} | X(p_X s_X)\rangle \langle X(p_X s_X)| j_{qb}^\mu | B(p_B)\rangle \quad (4)$$

must have the form

$$h^{\mu\nu} = -\alpha g^{\mu\nu} + \sum_{\sigma_1 = \pm, \sigma_2 = \pm} \beta_{\sigma_1\sigma_2}(p_B + \sigma_1 p_X)^\mu (p_B + \sigma_2 p_X)^\nu + i\gamma\epsilon^{\mu\nu\rho\sigma}(p_B + p_X)_\rho (p_B - p_X)_\sigma; \quad (5)$$

if the electron mass is neglected, then the differential semileptonic decay rate for $B^{0,-} \to X_q^{+,0}e^-\bar{\nu}_e$ depends only on α, β_{++}, and γ and is given by

$$\frac{d^2\Gamma}{dx\,dy} = |V_{qb}|^2 \frac{G_F^2 M_B^5}{32\pi^3}\left\{\frac{\alpha}{M_B^2}y + 2\beta_{++}\left[2x\left(1 - \frac{M_X^2}{M_B^2} + y\right) - 4x^2 - y\right] - \gamma y\left[1 - \frac{M_X^2}{M_B^2} - 4x + y\right]\right\}, \quad (6)$$

where $x \equiv E_e/M_B$ and $y \equiv t/M_B^2 = (p_B - p_X)^2/M_B^2$. Of course the result (6) holds for other $M \to M'e^-\bar{\nu}_e$ decays with the appropriate substitutions; for decays to $e^+\nu_e$ one must in addition reverse the sign of the term proportional to γ.

We have estimated α, β_{++}, and γ for various channels X using the quark model, building up the total electron spectrum $d\Gamma/dx$ by summing over contributing channels.[2] Our calculations were done in the nonrelativistic version of the quark potential model[3]; possible improvements on this simplest model are discussed in Ref. 2. One assumption of our calculation is that the creation of additional quark-antiquark pairs can be ignored, so that the sum over final hadronic states X is saturated by $q\bar{d}$ (or $q\bar{u}$) states. That is, we assumed that all multihadron final states result from the decay of resonances. For convenience we also worked in the narrow-resonance approximation, but expect this detail to be of very minor importance to $d\Gamma/dE_e$. Our calculations were truncated by including only the states $X = 1\,^1S_0$, $1\,^3S_1$, $1\,^3P_2$, $^3P_1$, $1\,^3P_0$, $1\,^1P_1$, $2\,^1S_0$, $2\,^3S_1$; this was done both for practical reasons and because at higher masses one would have to consider gluonic excitations of mesons.

With these approximations we only needed to calculate $\langle X(p_X s_X)|j^\mu|B(p_B)\rangle$ for X a state of the ordinary quark model. Our method[4] was to make a correspondence between the Lorentz-invariant form factors f which occur in the expansion of this matrix element and those (which we call $\tilde{f}$) which appear in the quark-model calculation of $\langle \tilde{X}(\tilde{p}_X\tilde{s}_x)|j^\mu|\tilde{B}(\tilde{p}_B)\rangle$, where $\tilde{M}$ is the weak-binding, nonrelativistic quark-model state corresponding to the meson M. Form factors $\tilde{f}$ which appear in terms that are of sufficiently low order in momenta can be calculated in the quark model. The corresponding form factors f are taken to match onto $\tilde{f}$ at the zero recoil point; higher-order form factors are neglected.

As an example consider $B^0 \to D^+$, where $B^0 = b\bar{d}$ $1\,^1S_0$ and $D^+ = c\bar{d}\,1\,^1S_0$. In general

$$\langle D^+(p_D)|j^\mu_{c\bar{d}}|B^0(p_B)\rangle = f_+(p_B+p_D)^\mu + f_-(p_B-p_D)^\mu, \quad (7)$$

where $f_\pm$ are Lorentz-invariant form factors which can depend on $t=(p_B-p_D)^2$. Since $\alpha=0$, $\beta_{++} = |f_+|^2$, and $\gamma=0$, only the f_+ form factor is actually required in this case. In the weak-binding, nonrelativistic limit, the matrix element of $j^\mu_{c\bar{d}}$ between $\tilde{B}^0(\tilde{p}_B)$ and $\tilde{D}^+(\tilde{p}_D)$ has exactly the form (7), with

$$(\tilde{M}_B+\tilde{M}_D)\tilde{f}_+ + (\tilde{M}_B-\tilde{M}_D)\tilde{f}_- = (4\tilde{M}_B\tilde{M}_D)^{1/2}\int d^3p\,\phi_D^*[\mathbf{p}+(m_d/\tilde{M}_D)\mathbf{p}_D]\phi_B(\mathbf{p}), \quad (8)$$

$$(\tilde{f}_+ - \tilde{f}_-)\mathbf{p}_D = (4\tilde{M}_B\tilde{M}_D)^{1/2}\int d^3p\,\phi_D^*[\mathbf{p}+(m_d/\tilde{M}_D)\mathbf{p}_D]\phi_B(\mathbf{p})[\mathbf{p}/2m_b+(\mathbf{p}+\mathbf{p}_D)/2m_q], \quad (9)$$

for $\mathbf{p}_B=0$ and $\mathbf{p}_D \ll \tilde{M}_D$; in Eqs. (8) and (9), $\phi(\mathbf{p})$ are the momentum-space wave functions. Our prescription is to take $f_\pm(t-t_m) = \tilde{f}_\pm(t-t_m)$, where $t_m=(M_B-M_D)^2$.

To complete these calculations we needed explicit meson wave functions.[3] We chose to use the Schrödinger wave functions appropriate to the usual Coulomb plus linear potential,

$$V(\mathbf{r}) = -4\alpha_s/3r + c + br, \quad (10)$$

with $\alpha_s = 0.5$ GeV, $c=-0.84$ GeV, and $b=0.18$ GeV^2, and with constituent-quark masses $m_u=m_d=0.33$ GeV, $m_s=0.55$ GeV, $m_c=1.82$ GeV, and $m_b=5.12$ GeV. This simplified model gives quite reasonable spin-averaged spectra of $u\bar{d}$, $c\bar{d}$, and $b\bar{d}$ mesons, and extends satisfactorily to the $c\bar{c}$ and $b\bar{b}$ systems (where we do not need it) with a running $\alpha_s=0.4$ and 0.3, respectively.

A detailed account of our formulas for α, β_{++}, and γ for the eight states X that we considered and of our meson wave functions is given in Ref. 2. Here we will simply show the resulting curves for $d\Gamma/dE_e$ for B decay. Before doing so, however, we must address a generic problem of our nonrelativistic calculation. The reader may already have wondered how we can, for example, calculate slopes of form factors, since such slopes appear in coefficients of $\mathbf{p}_X^2$ and so may contain contributions from relativistic effects. Indeed, the answer is that our effective radii r_f $[f(\mathbf{p}_X^2) = 1 - \frac{1}{6}r_f^2\mathbf{p}_X^2 + \ldots]$ include only wave-function-overlap effects and so can be in error by terms of order $1/m_q$. In a truly nonrelativistic situation $1/m_q \ll r_f$ so that such corrections are unimportant, but in the cases at hand we are not surprised to find that our calculated pion and kaon charge radii are about 30% too small. We have therefore compensated all of our effective radii by this factor to produce our best estimate of semileptonic decay rates. With a few exceptions, to be discussed below, whether or not this defect is corrected is of little importance.

In Ref. 2 we have compared our quark-model calculation of semileptonic decay rates with the known meson semileptonic decays $K \to \pi$ and $D \to X_s$. The comparison is satisfactory: both $f_+(t)$ and $f_-(t)$ in $K \to \pi$ are predicted within errors, and in $D \to X_s$ we predict correctly the $K(495)$:$K^*(895)$ ratio, the shape and magnitude of the total electron spectrum, and the $D \to K$ form factor.[5]

Figures 1 and 2 show our predictions for $B \to X_c$ and $B \to X_u$, respectively. In each case we have shown how the spectrum that we have calculated is built up out of exclusive final states, as well as the free-quark spectrum. In $B \to X_c$ we predict that the inclusive spectrum is nearly saturated by $D(1870)$ and $D^*(2020)$ production (representing respectively 19%

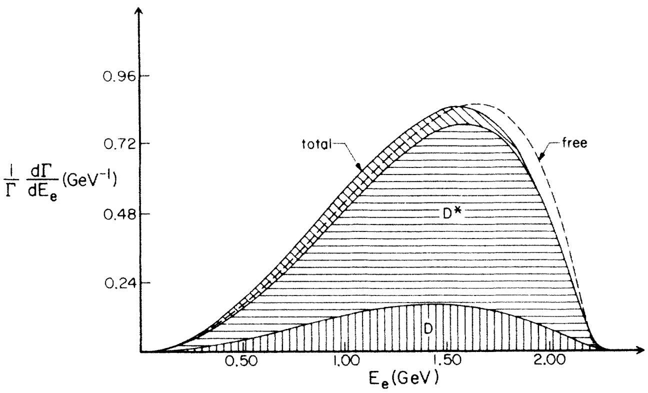

FIG. 1. $(1/\Gamma)d\Gamma/dE_e$ for $B \to X_c e^- \bar{\nu}_e$ showing the contributions of D and D^*, and the total contributions from $1S$, $1P$, and $2S$ states; also shown is the corresponding free-quark curve. Absolute rates can be obtained by use of $\Gamma = 0.58 \times 10^{14} |V_{cb}|^2 \text{ sec}^{-1}$ and $\Gamma^{\text{free}} = 0.49 \times 10^{14} |V_{cb}|^2 \text{ sec}^{-1}$.

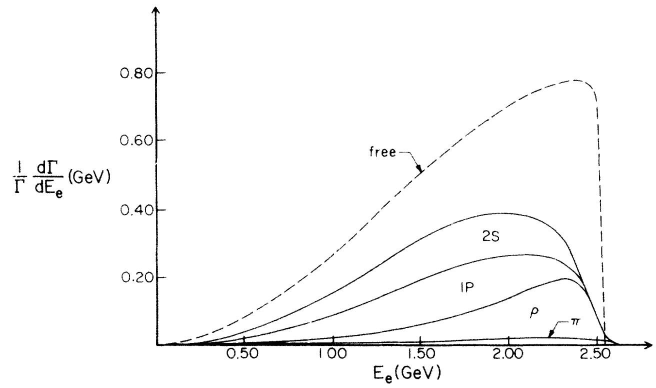

FIG. 2. $(1/\Gamma^{\text{free}})d\Gamma/dE_e$ for $B^0 \to X_u^+ e^- \bar{\nu}_e$ showing the contributions of π, ρ, the $1P$ states A_2, A_1, A_0, and B, and the $2S$ states π' and ρ'; also shown is the free-quark curve $(1/\Gamma^{\text{free}})d\Gamma^{\text{free}}/dE_e$. Absolute rates can be obtained by use of $\Gamma^{\text{free}} = 1.18 \times 10^{14} |V_{ub}|^2 \text{ sec}^{-1}$.

and 71% of the total). This is consistent with the experimental situation.[6] Our predicted total semileptonic rate is $0.58 \times 10^{14} |V_{cb}|^2 \text{ sec}^{-1}$ and leads us to extract from experiment[6] a value of $|V_{cb}| = 0.041 \pm 0.004 \pm 0.005$. We will discuss the uncertainties in this determination associated with our calculation (the second error quoted; the first error is experimental) below. Of crucial importance for experimental studies of the end-point region is that our spectrum, which is softer than the free-quark spectrum, gives a good fit to experiment with no $b \to u$ contribution.[2] To set a limit on (or, eventually, to determine) V_{ub} of course requires the use of the $B \to X_u$ electron spectrum. In this case one would not expect our calculation to have exhausted the inclusive spectrum, and this is consistent with the strong contribution of the radially excited states in this case. Our predicted $b \to u$ spectrum is now very much softer than the free-quark spectrum; this will in turn considerably soften limits on $|V_{ub}|^2/|V_{cb}|^2$ from the end-point spectrum.

From the success of our calculations for $K \to \pi$ and $D \to X_s$, and from studies of the sensitivity of our results to the quark-model wave functions, we believe that our predictions for $B \to X_c$ are quite reliable. In particular, variations of our wave functions over a wide range (encompassing, e.g., wave functions which have 30% larger radii) produce less than a 20% variation in our absolute predictions for $d\Gamma/dE_e$ and almost no variation in the shape of the spectrum. The same is true for our $K \to \pi$ and $D \to X_s$ predictions. (Note that our predictions for $B \to X_c$ can be further checked by study of the $B \to D$ and $B \to D^*$ components of the spectrum.) On the other hand, while the shape of our predicted $B \to X_u$ spectrum is also very stable, our absolute prediction of $d\Gamma/dE_e$ is in this case quite sensitive to our wave functions (or, e.g., to our form factor modifications). Thus while we believe that our $B \to X_u$ end-point spectrum prediction is the best one that can be made, we would assign a possible 50% error to the absolute normalization of our curve.

Most attempts to extract $|V_{ub}|^2/|V_{cb}|^2$ from the $B \to Xe^- \bar{\nu}_e$ end-point spectrum have fitted with the form (1), with the $d\hat{\Gamma}$'s given by a QCD-perturbed free-quark calculation in which extra parameters were introduced to correct for nonperturbative effects.[7] With recent improvements in the data, these attempts have, as might have been anticipated, encountered difficulties.[6] These difficulties have made it clear that the predicted end-point behavior of the Ref. 7 calculations was being controlled almost entirely not by perturbative QCD, but by the *ad hoc* parameters introduced to describe bound-state effects. Our calculation, in contrast, is especially suitable for the end-point region: Not only is the dynamics of the quark model more appropriate to this region, but also it correctly handles the kinematics of the opening of new channels with their appropriate quantum numbers. We have argued that it also predicts the $B \to X_c$ and $B \to X_u$ spectral shapes reliably, and so can be used to extract $|V_{cb}|^2$ and $|V_{ub}|^2$ with errors associated mainly with our predictions of the absolute normalization of these spectra.[2]

Our main conclusions are thus that the quark model can be used to make reasonably reliable predictions about the end-point region in semileptonic B decays which are dominated by nonperturbative QCD. These predictions indicate that the limits on $|V_{ub}|^2/|V_{cb}|^2$ will be weaker than had been thought. As a balance to this somewhat discouraging result is our prediction of the partial rates to exclusive $B \to X_u$ channels which suggest the possibility of determining $|V_{ub}|$ from studies of exclusive modes like $B^- \to \rho^0 e^- \bar{\nu}_e$. Finally, we note that as byproducts of this calculation we have made detailed predictions[2] of how the $B \to X_c$,

$B \rightarrow X_u$, $D \rightarrow X_s$, and $D \rightarrow X_d$ spectra are built up out of exclusive channels.[8] One application of these latter predictions could be the extraction of $|V_{cd}|$ from an exclusive mode like $D^+ \rightarrow \rho^0 e^+ \nu_e$ to produce an independent check on the validity of the KM parametrization.

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