Spin-Dominated Inflation in the Einstein-Cartan Theory

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In the framework of the simplest Einstein-Cartan generalization of the standard cosmological model, it is shown that an inflationary phase may occur at a sufficiently early epoch, during which the dominant contributions to the effective energy-momentum tensor are represented by the spin density of the matter sources. The general features of this inflationary scenario are briefly discussed, and in particular the possibility of obtaining an expansion rate different from that of the standard exponential inflation is stressed.

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It is well known that, according to the standard four-dimensional cosmological model, a suitable inflationary phase of accelerated expansion[1] can occur only during a period of vacuum dominance. In fact, in the case of a homogeneous and isotropic universe filled with an ideal fluid with pressure p and energy density ρ, the Robertson-Walker scale factor $R(t)$ satisfies the Friedman equation

$$\ddot{R} = -\tfrac{4}{3}\pi G(\rho+3p)$$

(the dot denotes the derivative with respect to the cosmic time t); the condition $\ddot{R} > 0$ (required to solve the horizon, flatness, and entropy problems[2]) implies then the appearance of a negative effective pressure, $p < -\rho/3$, which can only be achieved, in this scenario, with a dominant vacuum contribution to the total stress-energy tensor.[3]

The main object of this paper is to point out that if the spin contributions of the matter sources are included in the gravitational field equations according to the Einstein-Cartan theory,[4] then a period of effective inflation ($\ddot{R} > 0$) can occur at a sufficiently early epoch when the energy content of the universe is spin dominated. In this case the inflation is driven by the temporal evolution of the spin-density tensor.

In the Einstein-Cartan theory, which is the simplest example of Poincaré gauge theory of gravity,[4] the gravitational Lagrangean is the usual scalar curvature, $L_g = (-g)^{1/2}R(\Gamma)$, constructed, however, with a connection Γ which is metric compatible but nonsymmetric; that is $\Gamma_{\alpha\beta}{}^{\mu} = \{{}_{\alpha}{}^{\mu}{}_{\beta}\} - K_{\alpha\beta}{}^{\mu}$, where $\{{}_{\alpha}{}^{\mu}{}_{\beta}\}$ is the Christoffel connection and $K_{\alpha\beta}{}^{\mu}$ the contorsion tensor, related to the torsion $Q_{\alpha\beta}{}^{\mu} = \Gamma_{[\alpha\beta]}{}^{\mu}$ by $K_{\alpha\beta}{}^{\mu} = -Q_{\alpha\beta}{}^{\mu} - Q^{\mu}{}_{\alpha\beta} + Q_{\beta}{}^{\mu}{}_{\alpha}$. Starting then from the total Lagrangean $L_g + L_m$, with L_m representing the matter sources, and performing the variation with respect to the metric $g_{\alpha\beta}$ and the contorsion $K_{\alpha\beta\mu}$, one is led to the following field equations[4]:

$$G^{\alpha\beta} - (\nabla_{\mu} + 2Q_{\mu\nu}{}^{\nu})(T^{\alpha\beta\mu} - T^{\beta\mu\alpha} + T^{\mu\alpha\beta}) = \chi T^{\alpha\beta}, \quad (1)$$

$$T^{\alpha\beta\mu} = \chi\tau^{\alpha\beta\mu}, \quad (2)$$

where $\chi = 8\pi G$ is the Newton coupling constant ($c=1$),

$$T_{\alpha\beta}{}^{\mu} = Q_{\alpha\beta}{}^{\mu} + \delta^{\mu}_{\alpha}Q_{\beta\nu}{}^{\nu} - \delta^{\mu}_{\beta}Q_{\alpha\nu}{}^{\nu},$$

and $G^{\alpha\beta}$ and ∇_{μ} are respectively the Einstein tensor and the covariant derivative for the full nonsymmetric connection Γ; finally

$$T^{\alpha\beta} = \frac{2}{(-g)^{1/2}}\frac{\delta L_m}{\delta g_{\alpha\beta}}, \quad \tau^{\alpha\beta\mu} = \frac{1}{(-g)^{1/2}}\frac{\delta L_m}{\delta K_{\mu\beta\alpha}} \quad (3)$$

are respectively the dynamical (symmetric) energy-momentum tensor and the canonical spin-density tensor.

As torsion is algebraically related to the matter spin density, one can substitute spin for torsion everywhere in Eq. (1), using Eq. (2), to obtain the following modified Einstein equations[5]:

$$G^{\alpha\beta}(\{\,\}) = \chi(T^{\alpha\beta} + \tau^{\alpha\beta}), \quad (4)$$

where $G^{\alpha\beta}(\{\,\})$ is the usual symmetric Einstein tensor for the Christoffel connection, and

$$\tau^{\alpha\beta} = \chi[-4\tau^{\alpha\mu}{}_{[\nu}\tau^{\beta\nu}{}_{\mu]} - 2\tau^{\alpha\mu\nu}\tau^{\beta}{}_{\mu\nu} + \tau^{\mu\nu\alpha}\tau_{\mu\nu}{}^{\beta} + \tfrac{1}{2}g^{\alpha\beta}(4\tau_{\lambda}{}^{\mu}{}_{[\nu}\tau^{\lambda\nu}{}_{\mu]} + \tau^{\mu\nu\lambda}\tau_{\mu\nu\lambda})] \quad (5)$$

represents the contribution of an effective spin-spin interaction induced by the generalized geometric structure of the theory.

Since in this paper we are interested in an Einstein-Cartan generalization of the standard cosmological model (in which matter is represented as an ideal fluid), we suppose that L_m describes a spinning fluid minimally coupled to the metric and the torsion of the Riemann-Cartan manifold. Using then the variational formalism recently developed by Ray and Smalley,[6] one obtains that the canonical spin tensor is given by

$$\tau^{\alpha\beta\mu}=\tfrac{1}{2}S^{\alpha\beta}u^{\mu} \tag{6}$$

($S_{\alpha\beta}$ is the spin density and u^{μ} the four-velocity of the fluid), and the dynamical energy-momentum can be decomposed into the usual perfect fluid part, $T_F{}^{\alpha\beta}$, and an intrinsic-spin part $T_S{}^{\alpha\beta}$, that is $T^{\alpha\beta}=T_F{}^{\alpha\beta}+T_S{}^{\alpha\beta}$, where, with the torsion contributions written explicitly,

$$T_S{}^{\alpha\beta}=u^{(\alpha}S^{\beta)\mu}u^{\nu}u_{\mu;\nu}+(u^{(\alpha}S^{\beta)\mu})_{;\mu}+Q_{\mu\nu}{}^{(\alpha}u^{\beta)}S^{\nu\mu}-u^{\nu}S^{\mu(\beta}Q^{\alpha)}{}_{\mu\nu}-\omega^{\mu(\alpha}S^{\beta)}{}_{\mu}+u^{(\alpha}S^{\beta)\mu}\omega_{\mu\nu}u^{\nu} \tag{7}$$

(ω is the angular velocity associated with the intrinsic spin,[6] and a semicolon denotes the usual Riemann-Christoffel covariant derivative).

It should be noted that this "improved" energy-momentum tensor differs from that phenomenologically assumed in the context of a Weyssenhoff semiclassical model of spinning fluid,[7] and the correction terms in Eq. (7) are due to the treatment of spin as a thermodynamical variable; the Weyssenhoff convective condition $S_{\alpha\beta}u^{\beta}=0$, however, continues to hold also in this variational formalism.[6]

According to the usual interpretation of the Einstein-Cartan theory, we assume in this paper that $S_{\alpha\beta}$ is associated with the quantum-mechanical spin of microscopic particles[4,5]; the effective sources of the macroscopic gravitational field are to be defined then by a suitable space-time averaging[5] of the tensors $T^{\alpha\beta}+\tau^{\alpha\beta}$ (which describe the matter sources in the microscopic domain). In this case, it is important to stress that even if the spins are randomly oriented, the average of the spin-squared terms is not vanishing in general,[5] so that the Einstein-Cartan field equations are different from the general-relativistic ones even in the classical macroscopic limit. For an unpolarized spinning field, in particular, the averaging procedure gives, if we put $\langle S_{\alpha\beta}\rangle=0$ and $\sigma^2=\frac{1}{2}\langle S_{\alpha\beta}S^{\alpha\beta}\rangle$ in Eq. (5),

$$\langle\tau^{\alpha\beta}\rangle=\tfrac{1}{2}\chi\sigma^2u^{\alpha}u^{\beta}+\tfrac{1}{4}\chi\sigma^2g^{\alpha\beta}. \tag{8}$$

Moreover, $\langle T_F{}^{\alpha\beta}\rangle=(\rho+p)u^{\alpha}u^{\beta}-pg^{\alpha\beta}$ and, using the definition of ω in terms of the orthonormal tetrads $a_{\mu}^{(i)}$ (see Ref. 6), in the case of randomly oriented spins one has $\langle a_{\mu}^{(i)}\rangle=0$ for $i=1,2,3$, and $\langle a_{\mu}^{(4)}\rangle=u_{\mu}$, so that $\langle\omega_{\alpha\beta}\rangle=u^{\mu}(\nabla_{\mu}u_{\alpha})u_{\beta}$. Equation (7) gives then

$$\langle T_s{}^{\alpha\beta}\rangle=-\chi\sigma^2u^{\alpha}u^{\beta}. \tag{9}$$

(Note that, in this averaged limit, the intrinsic-spin part of the dynamical energy-momentum tensor of Ray and Smalley[6] coincides exactly with that which one would obtain by averaging the corresponding phenomenological expression in the semiclassical Weyssenhoff model; see for example, Nurgaliev and Ponomariev.[8])

The simplest Einstein-Cartan generalization of the standard cosmological scenario is obtained then, considering the universe filled with an unpolarized spinning fluid, and solving the modified Einstein equations $G^{\alpha\beta}(\{\,\})=\chi\theta^{\alpha\beta}$, where $\theta^{\alpha\beta}$ describes the effective gravitational sources in the macroscopic limit, i.e.,

$$\theta^{\alpha\beta}=\langle T^{\alpha\beta}\rangle+\langle\tau^{\alpha\beta}\rangle$$
$$=(\rho+p-\tfrac{1}{2}\chi\sigma^2)u^{\alpha}u^{\beta}-(p-\tfrac{1}{4}\chi\sigma^2)g^{\alpha\beta}. \tag{10}$$

A static cosmological solution corresponding to this generalized energy-momentum tensor has been recently investigated in Ref. 8. In this paper we assume that p, ρ, σ depend only on time, and that the universe is spatially homogeneous and isotropic, described by the Robertson-Walker metric. In a comoving frame, where $u^{\mu}=(0,0,0,1)$, we are led then to the following modified field equations for the scale factor (for simplicity we consider here a vanishing spatial curvature):

$$\ddot{R}=-\tfrac{4}{3}\pi GR(\rho+3p-8\pi G\sigma^2), \tag{11}$$

$$\dot{R}^2=\tfrac{8}{3}\pi GR^2(\rho-2\pi G\sigma^2), \tag{12}$$

and their combination gives

$$\frac{d}{dt}(\rho-2\pi G\sigma^2)=-3\frac{\dot{R}}{R}(\rho+p-4\pi G\sigma^2), \tag{13}$$

which generalizes the usual covariant energy conservation law to include the spin contributions.

Considering Eq. (11), one immediately obtains that an accelerated expansion ($\ddot{R}>0$) can be arranged even in the case of positive pressure, provided that the condition

$$8\pi G\sigma^2>\rho+3p \tag{14}$$

is satisfied.

In order to discuss the general features of this spin-dominated inflationary scenario, we suppose that, during the epoch in which the spin-squared corrections to Eqs. (11)–(13) are not negligible, matter can be described as a liquid of unpolarized fermions with spin $\hbar/2$, and we assume, as in Ref. 8, the equation of state $p=k\rho$ ($k<1$ to avoid that the speed of sound becomes greater than c). We have then $\sigma^2=\frac{1}{2}\langle S^2\rangle=\hbar^2\langle n^2\rangle/8$, where n is the particle number density, and the averaging procedure gives[8]

$$\sigma^2=\tfrac{1}{8}\hbar^2A_k{}^{-2/(1+k)}\rho^{2/(1+k)}, \tag{15}$$

where A_k is a dimensional constant depending on k (note, incidentally, that if $\langle S\rangle=0$ then $\langle S^2\rangle$ is just the square of the dispersion of the spin-density distribution around its average value, $\Delta S^2 = \langle S^2 \rangle - \langle S \rangle^2 = \langle S^2 \rangle$). In this case the integration of the conservation law (13) can be easily performed, and one obtains

$$\rho = aR^{-3(1+k)}, \tag{16}$$

where a is an integration constant.

It is important to stress that, in this model of the universe, the spin contributions to the field equations play the role of "centrifugal forces," and produce a bouncing which avoids the initial singularity. The temporal evolution of this model is characterized then by a maximal initial density, and in fact from Eq. (12) one obtains $\rho < \rho_i$, where

$$\rho_i = \left[\frac{4A_k^{2/(1+k)}}{\pi G \hbar^2} \right]^{(1+k)/(1-k)}. \tag{17}$$

The corresponding minimal initial value R_i of the scale factor is then, from (16),

$$R_i = a^{1/3(1+k)} \left[\frac{\pi G \hbar^2}{4A_k^{2/(1+k)}} \right]^{1/3(1-k)}. \tag{18}$$

(Note that for $k=0$ one obtains from these expressions the minimal radius and the maximal density deduced by Kopczynski[9] and Trautman[10] in the framework of a model of universe filled with polarized dust.)

As the condition (14) defining the inflationary phase can be rewritten $\rho > \rho_f$, where

$$\rho_f = \left[\frac{(1+3k) A_k^{2/(1+k)}}{\pi G \hbar^2} \right]^{(1+k)/(1-k)}, \tag{19}$$

it follows that as soon as the universe begins to expand, starting with an initial density ρ_i and radius R_i, its acceleration is positive ($\ddot{R} > 0$), since $\rho_i > \rho_f$, so that the earliest evolutionary stages of the universe, according to this simple model, are characterized by a spin-dominated inflationary expansion. The inflation stops when the density ρ_f is reached, corresponding to a scale factor

$$R_f = a^{1/3(1+k)} \left[\frac{\pi G \hbar^2}{(1+3k) A_k^{2/(1+k)}} \right]^{1/3(1-k)}, \tag{20}$$

and after that time the acceleration becomes negative and the spin contributions negligible, so that the expansion follows the usual Friedman behavior.

It is interesting to observe that, as the density ranges from ρ_i to ρ_f, according to this scenario the universe goes through three different types of inflationary phases. Consider in fact the Hubble parameter $H = \dot{R}/R$: From the field equations (11) and (12) one has

$$\dot{H} = -4\pi G(\rho + p - 4\pi G \sigma^2) \tag{21}$$

and then $\dot{H} = 0$ for $\rho = \rho_c$, where

$$\rho_c = \left[\frac{2(1+k) A_k^{2/(1+k)}}{\pi G \hbar^2} \right]^{(1+k)/(1-k)} \tag{22}$$

(note that $\rho_i > \rho_c > \rho_f$). At the beginning of the expansion we have $\rho > \rho_c$, so that the inflation is characterized by $\ddot{R} > 0$, $\dot{H} > 0$: This phase, called "superinflation," has been recently discussed by Lucchin and Matarrese.[11] When the density reaches ρ_c, we have $\ddot{R} > 0$, $\dot{H} = 0$, and this corresponds to an exponential inflation, like that originally considered by Guth.[3] Finally, for $\rho_c > \rho > \rho_f$, the expansion satisfies $\ddot{R} > 0$, $\dot{H} < 0$, so that the universe goes through a phase of power-law inflation.[2,11]

In conclusion, it must be remarked that a physically interesting inflationary scenario should be characterized by an inflation factor $Z = R_f/R_i$ sufficiently high to be able to solve the problems of the standard model,[3] i.e., $Z \gtrsim 10^{30}$. For the simplistic model considered here this condition becomes $4/(1+3k) \gtrsim 10^{90(1-k)}$, and obviously it cannot be satisfied, unless negative values of k are allowed. In this case, it is interesting to observe that unlike in the standard inflationary scenario, a large amount of inflation can be achieved even for $k > -\frac{1}{3}$; this requires, however, an extreme fine tuning of the parameter k: In fact, putting $k = -\frac{1}{3} + \epsilon$, with $\epsilon > 0$, we obtain $Z \geqslant 10^{30}$ for $\epsilon < 10^{-120}$. (Moreover, $k < 0$ requires a substantial contribution from the vacuum energy density, even though, for $-\frac{1}{3} < k < 0$, a sort of "vacuum with spin"—see for example Soffel, Müller, and Greiner[12]—is involved.)

It is possible, however, that this difficulty disappears in the framework of a more realistic mechanism of spin-dominated inflation, based on cosmological models in which spinning matter is polarized[13] (for example, because of the presence of a primeval magnetic field[10,14]), or torsion is propagating, like in the case of a Poincaré gauge theory of gravity (for a reveiw see Hehl[15] and Hayashi and Shirafuji[16]).

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