High-Dimension Chaotic Attractors of a Nonlinear Ring Cavity

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The route to chaos found from the delay-differential rate equations is basically different from the subharmonic cascade which is generic of the adiabatic-following-approximation equation. The Lyapunov dimension of each chaotic attractor is found to increase linearly with $\gamma\tau_R$, the ratio of the delay time to the medium lifetime. This clearly shows the *invalidity* of the difference-equation mapping whose dimension never exceeds 2.

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In a well-known paper, Ikeda[1] predicted that a plane-wave light beam transmitted by a ring cavity containing resonant two-level atoms can exhibit a subharmonic cascade terminating on a nice spirallike strange attractor with a low fractal dimension. Numerical studies[2] of the Ikeda model displayed period-doubling sequences which were found to be consistent with the conjecture of universality.[3] Such a bifurcation sequence follows from a two-dimensional (2D) difference equation, which is an approximation of the ring-cavity delay-differential rate equation. The periodic output intensities consist of trains of square pulses with fundamental period equal to twice the round-trip time τ_R. No transient oscillation between time intervals τ_R is taken into account, according to the adiabatic-following approximation[4] required for the Ikeda model to be valid. We show that the adiabatic approximation leads to erroneous conclusions about the route to chaos and the dimension of the attractors.

The domain of validity of the adiabatic-following approximation can be easily checked in standard nonlinear-optics experiments that have a single pass of the light through the nonlinear medium because all initial data are known. It becomes much more difficult for multiple-pass experiments because transients interact repeatedly with the atoms and spread out over the full interval τ_R. If any transient interaction lasts an atomic relaxation time $T_1=\gamma^{-1}$, then the number of degrees of freedom is roughly equal to the number $\gamma\tau_R$ of cavity modes which might be excited within the linewidth γ and thus increases linearly with τ_R. Such a description disagrees with the usual feeling that the Ikeda mapping (with two degrees of freedom) works in the limit $\gamma\tau_R \gg 1$. Indeed we find that the route to chaos is very different from the subharmonic one[5,6] and the fundamental oscillation is not a square-wave form.

The delay-differential equation is an infinite-dimensional dynamic system; such systems often exhibit finite-dimension attractors.[7] Farmer shows that the attractors of the Mackey-Glass delay-differential equation have finite dimensions that increase linearly with τ_R beyond the periodic windows.[7] A similar linear growth with space dimension was also found in a partial differential equation.[8] Nonetheless, no generic route to chaos in delay-differential systems is known. Might the $2\tau_R$ period-doubling sequence be generic as predicted in the 2D model? Might the higher-harmonic frequency locking, as observed[5] and numerically found[6] in a hybrid device, be generic? Is the linear increase of the attractor dimension with τ_R a general property of delay-differential systems? To address these questions, we have (a) numerically integrated the plane-wave ring-cavity delay-differential rate equation in the limit $\gamma\tau_R \gg 1$ and studied the bifurcation sequence as a function of the input intensity and compared it to the 2D mapping predictions[1,2] (Figs. 1 and 2); and (b) studied the attractors for a given input intensity when $\gamma\tau_R$ is varied (Fig. 3).

The ring-cavity delay-differential rate equations are

$$E(t)=E_0+RE(t-\tau_R)\times\exp\{i\chi[\phi(t-\tau_R)-1]\} \tag{1}$$

and, with $\phi(0)=0$,

$$\dot{\phi}=d\phi(t)/dt=-\gamma[\phi(t)+A(\phi)|E(t)|^2]. \tag{2}$$

$E(t)$, the complex field amplitude at the entrance to the cell containing two-level atoms, is scaled to the square root of the atomic saturation intensity. Equation (1) gives the boundary condition: $E(t)$ at time t is the sum of the intracavity input amplitude E_0 and the cell-exit field amplitude at time $t-\tau_R$ after it has been reflected by the output and input mirrors with intensity reflectivity $R<1$ and the feedback mirrors with reflectivity 1; $T=1-R$. The phase factor $\phi(t)$ measures the amount of energy absorbed by the cell, and $\chi=\alpha l$ is the *linear* susceptibility, where α^{-1} and l are the off-resonance absorption length and the cell length, respectively, and $\Delta=2\pi T_2(\nu_{\rm at}-\nu_f)$ is the detuning between the atomic and light-beam frequencies, scaled to the homogeneous halfwidth $(2\pi T_2)^{-1}$. In the limit $\dot{\phi}=0$, the 2D mapping equation is recovered. Here we keep the exact expression for $A(\phi)$,

$$A(\phi)=[\exp\{\alpha l(\phi-1)\}-1]/\alpha l, \tag{3}$$

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because the "dispersive limit" generally assumed by previous workers[1,2] is not always valid.

Figure 1 displays numerically computed time dependences and phase-space attractors (after the transient regime) for R, αl, and Δ values used previously by Ikeda[1a] and Carmichael.[2] The Runge-Kutta algorithm was used, and $\Delta t/T_1$ was varied from 0.05 to 0.005; for $\Delta t/T_1 \lesssim 0.01$, no change in the spectra or attractors was found for all E_0 considered. From the fixed point for $E_0 \lesssim 1.03$, the system bifurcates on a limit cycle with period $2.3\tau_R$ stable for E_0 up to 1.07. No period doubling (or frequency locking) was found, although we carefully looked for it using increments of 10^{-3} for E_0 from 1.065 to 1.08, i.e., increments about 8 times smaller than the expected domain for the period-doubled wave form. The high-frequency variations in Figs. 1(a)–1(c) may be responsible for chaos and termination of the period-doubling sequence. For higher E_0, a quasiperiodic window with two periods of order $2\tau_R$ and $3\tau_R$ appears. Note that in the limit of very high E_0, the system terminates on a fixed point with $E(t) \rightarrow E_0$ and $\phi(t) \rightarrow 1$. This bifurcation sequence can be compared with that of the 2D limit displayed in Fig. 2. The fundamental limit cycle occurs at the same E_0 and the $E_0 = 1.26$ wave forms are quite similar, but period doubling to chaos occurs only for the 2D map. The adiabatic-following approximation, $\phi = 0$, clearly breaks down, even when the actual limit cycle has a period close to $2\tau_R$ and the actual chaotic attractor has a spiral shape: Compare Fig. 1(e) with Fig. 3(d).

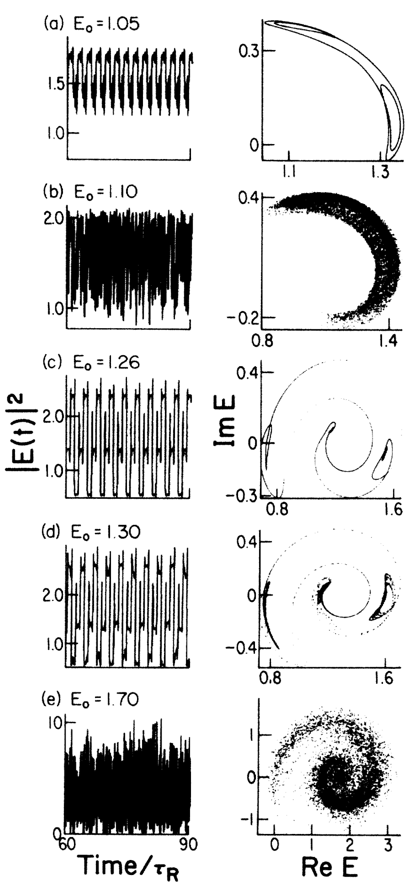

FIG. 1. Numerical integration of Eqs. (1), (2), and (3) with $R = 0.95$, $\alpha l = 4$, $\Delta = 3\pi$, and $\gamma\tau_R = 10$. No period-doubled wave form was found between (a) and (b) for 10^{-3} steps in E_0 between 1.065 and 1.08.

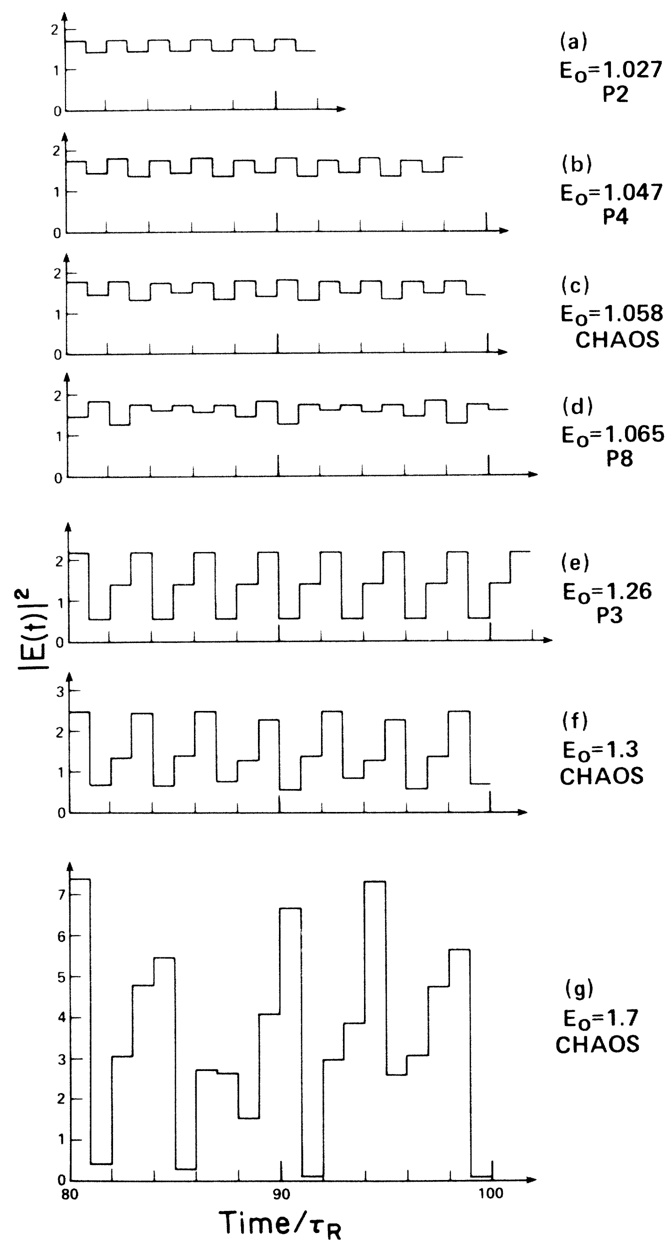

FIG. 2. Bifurcation sequence for the 2D mapping with same parameters as Fig. 1. A period-doubling route to chaos occurs between (b) and (c). Periodic windows with the chaotic domain also occur; see (d) and (e).

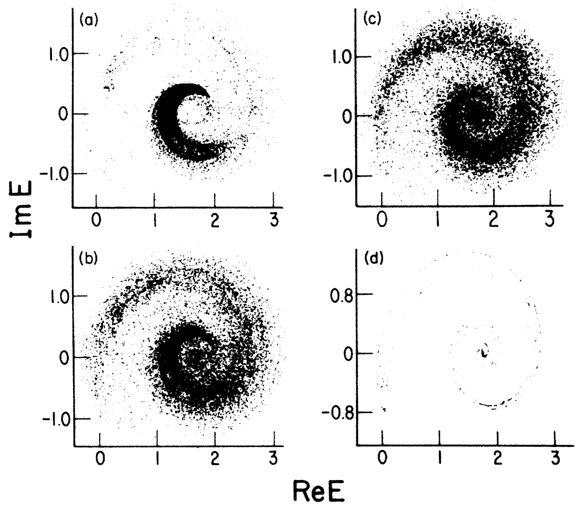

FIG. 3. Chaotic attractors as a function of $\gamma\tau_R$ for $E_0 = 1.7$ and other parameters the same as in Fig. 1. (a) $\gamma\tau_R = 1$, (b) $\gamma\tau_R = 2$, and (c) $\gamma\tau_R = 5$, (d) "$\gamma\tau_R = \infty$" or 2D mapping approximation.

It has been noted that the $2\tau_R$ oscillation and the period-doubling cascade are promoted when the initial (low E_0) phase $\Delta\alpha l/2$ is an odd multiple of π.[9] This is verified for the ring-cavity dynamical system: The choice of an odd multiple, if large enough, gives rise to a period doubling.[10] Nevertheless, the subharmonic cascade is thwarted and finally interrupted by harmonics.

Numerical integration of Eqs. (1) and (2) with $\exp(\alpha l\phi) = 1$ shows that the dispersive-limit bifurcation sequence is quite different. Period doubling is obtained without the requirement $\Delta\alpha l/2 = (2k+1)\pi$, and no periodic window is seen. Thus the nonlinear absorption is generally *not* negligible even for input intensities small compared with αl. Approximations valid for a single pass break down, *a priori*, for multiple passes.

In summary, our numerical studies show that a subharmonic sequence is not generic for a ring-cavity system. Another manifestation of the breakdown of the adiabatic-following approximation is the high dimensionality of the attractors.

The ergodic theory[11] of chaos provides quantities such as the Lyapunov exponents, the entropy, and the Hausdorff dimension to characterize the chaotic attractors. The positive (negative) exponents measure average exponential divergence (convergence) of nearby trajectories, onto the phase-space attractor. The metric entropy, h, is defined as the sum of the *positive* Lyapunov exponents Λ_i^+. It measures the rate of new information created by the dynamical system.

Here, the dimension D_L of an attractor is calculated by use of the conjecture of Kaplan and Yorke, which relates the dimension to the Lyapunov exponents set

TABLE I. Number of positive Lyapunov exponents N^+, the Lyapunov dimension D_L, the metric entropy h, and other parameters as in Fig. 1.

$\gamma\tau_R$	N^+	D_L	h
1	2	3.78	1.14
2	4	7.576	1.1
3	6	11.425	1.1
4	8	15.227	1.12
5	10	18.84	1.11
10	19	37.84	1.13

in decreasing order:

$$D_L = j + |\Lambda_{j+1}|^{-1} \sum_{i=1}^{j} \Lambda_i,$$

where j is the largest integer for which $\Lambda_1 + \ldots + \Lambda_j \geq 0$. The conjecture that the Lyapunov dimension D_L is equal to the information dimension has been verified by Farmer[7] for the delay-differential Mackey-Glass equation. The Lyapunov exponents are calculated numerically from Farmer's technique. Table I gives the number, N^+, of positive Lyapunov exponents and both h and D_L. For $\gamma\tau_R > 2$, the chaotic attractors look the same (see Fig. 3). Remarkable features appear in Table I. Both N^+ and D_L increase almost *linearly* with τ_R whereas the metric entropy is approximately constant. The linear increase of N^+ and D_L vs $\gamma\tau_R$ invalidates the adiabatic-following approximation for $\gamma\tau_R \to \infty$.[12] Indeed the dimension actually goes to infinity as $\gamma\tau_R$ does, whereas the dimension the of the Ikeda-mapping attractor can never exceed 2. Farmer's findings on the Mackey-Glass equation and the results here for the nonlinear ring cavity with nonlinear absorption suggest that a linear increase in dimension with increasing $\gamma\tau_R$ may be a universal feature of delay-differential nonlinear systems. The high dimensionality appears to arise from inherent fluctuations in the delay-differential equations as discussed by Pomeau *et al.*[13]

In summary, computations including the infinite-dimensional time variable show that even for large $\gamma\tau_R$, the ring-cavity route to chaos and the attractor's Lyapunov dimension differ markedly from those of the difference-equation map. Thus the success of the map in describing many of the experimental (continuous-time) observations using a hybrid device[5] must be considered fortuitous. The route to chaos of ring-cavity[14] and single-feedback-mirror[15] systems (analyzed thus far including the infinite-dimensional radial variable but neglecting the infinite-dimensional temporal variable) should be reexamined to determine the effect (possibly large) of the time derivative.

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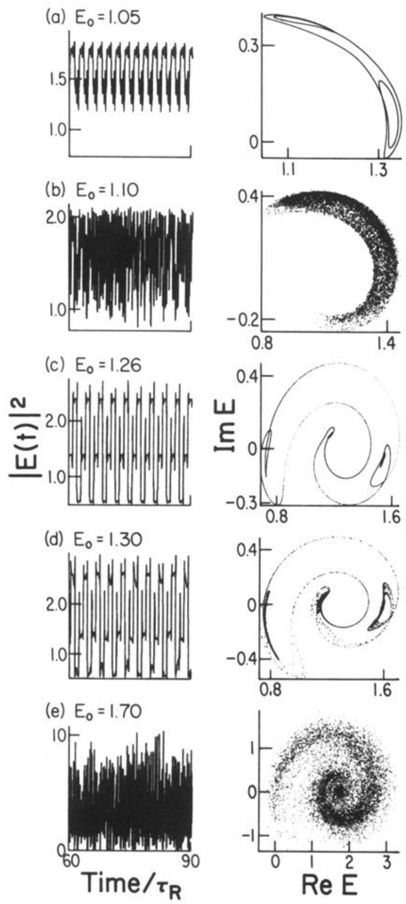

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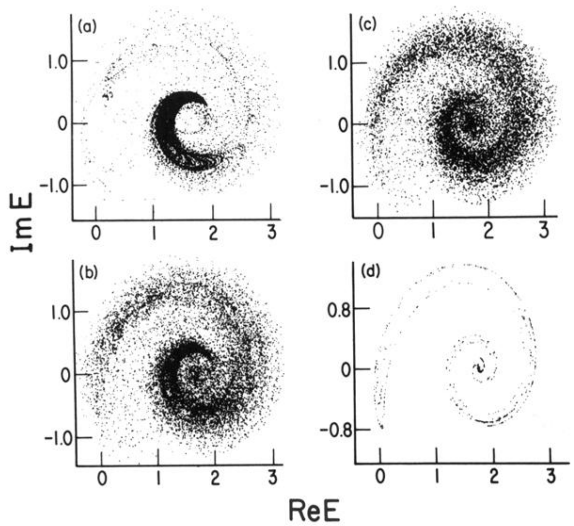

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