Cooperative Ring Exchange and the Fractional Quantized Hall Effect in an Incompressible Fluid

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We show that the cooperative-ring-exchange phenomenon and the consequent fractional quantization as suggested by Kivelson *et al.* can persist in a quantum fluid which is incompressible, like the Laughlin state. Simple arguments are given to suggest that the very existenct of the cooperative ring exchange may imply a melting instability of the triangular Wigner solid towards an incompressible fluid.

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It has been realized that the two-dimensional electrons in very strong magnetic fields exhibiting the fractional quantized Hall effect[1] have very unusual correlations. This is explicit in the Laughlin wave function and the associated quasiparticle properties.[2] But the physics behind the correlation and the origin of the fractional quantization has remained most unclear. Recently Kivelson, Kallin, Arovas, and Schrieffer[3] (KKAS) have argued that a cooperative ring exchange (CRE) is present in this system and that the fractional quantization could be explained on this basis. This fascinating proposal, on the face of it, seems to demand that the ground state should be a triangular-lattice Wigner solid. On the other hand, the Laughlin wave function, which is believed to contain the essential physics and which is quantitatively and qualitatively[4] so successful, describes an incompressible quantum fluid at the densities of interest. It is not clear if the real ground state is a liquid or not.[5]

In an attempt to understand the situation, I recently analyzed the Laughlin wave function and found[6] that the special correlation in it implies a dominant presence of CRE fluctuations. Motivated by this result and the compelling physics of KKAS theory I attempt to show in this paper that the KKAS theory is flexible and that the long-range spatial order is not a prerequisite. The only requirement is that the fluid should be incompressible and should have a triangular-lattice short-range correlation. I also give simple arguments to suggest that the presence of CRE implies an inherent melting instability of the triangular Wigner solid towards an incompressible fluid. The incompressible-liquid-like intermediate configurations in the path-integral formalism result in an annealed random-(imaginary-)field discrete Gaussian model. The random fields have long-range spatial correlation. The annealed model still exhibits the roughening transition and the consequent CRE and density quantization.

The theory of KKAS involves the evaluation of the partition function by a path-integral method. Special types of classical paths which correspond to the cooperative tunneling of rings of electrons in the Wigner lattice to permuted configurations are assumed to dominate the path integral. These "instantons" have a characteristic size τ_0. All the contributions of the ring exchanges happening in a time interval τ_0 are summed by modeling the change in the action by a discrete Gaussian model in an imaginary field,

$$H_{\mathrm{DG}} = \alpha(\nu) \sum_{\langle\lambda\mu\rangle} (S_\lambda - S_\mu)^2 + ih(\nu) \sum S_\lambda, \qquad (1)$$

where S_λ is an integer variable associated to every triangle in the lattice (or associated to the sites of the dual lattice which is a honeycomb lattice). S_λ counts the number of clockwise minus counterclockwise ring exchanges that surround a plaquette λ. The filling fraction is $\nu \equiv 1/m$. The function $\alpha(\nu)$ is a measure of the tunneling barrier. The function $h(\nu) = \pi(\nu^{-1} - 1)$ is the phase factor which arises as a result of (i) the magnetic flux enclosed by the exchange rings, (ii) the Fermi nature of the electrons, and (iii) the "complex" tunneling paths. The final partition function is assumed to be a product of the partition functions of discrete Gaussian models arising from every time slice τ_0.

First, I give a simple argument to suggest that the existence of CRE may imply a melting instability of the Wigner solid. The "thermal average" $L \equiv \langle (S_\lambda - S_\mu)^2 \rangle$ is a measure of how much a particular bond (shared by the nearest-neighbor plaquettes λ and μ) is involved in the CRE in a time interval τ_0. This is easily evaluated by use of the pair approximation used by Weeks and Gilmer.[7] For $m=3$, we are in the rough phase (CRE phase) and $L \approx 0.9$. This means that with very high probability the electrons at the end of the bond are away from their home sites (and gone into the complex plane) during the time interval τ_0. This is also consistent with the observation made by Kivelson.[8] It should be mentioned that L is model dependent and may get reduced if one models the short-range interaction between the exchange loops more accurately. However, a significant reduction seems unlikely because the existence of roughening may require an appreciable value of L.

Having observed a large value of L in the discrete Gaussian model, we will go back and see what it means to the various paths and the action in the path integral. If we use the quantity L as the Lindeman ratio used in the melting of classical solids we are far into the liquid region. However, this cannot be naively used here. The particles are not executing almost independent thermal motions as in a classical solid. The dynamics of the present problem is governed by a Hamiltonian with only first-order time derivatives, which give rise to its own peculiar properties. Motivated by the large Lindeman ratio, I give an alternative argument. If we consider a rigid Wigner solid and allow one line of atoms to tunnel coherently they see a potential which is periodic with the periodicity of the lattice. We saw before that every electron, including those in the neighborhood of the chain, participates in the tunneling with probability of the order of unity even in a short time interval τ_0. Thus, if we observe the coherent motion of one chain over a time long compared to τ_0, the potential that it sees will not be periodic. The physically important rings being one dimensional and long, this can result in the destruction of the long-range order along the chain rather easily. This in turn will feed back and affect the rest of the neighborhood, resulting possibly in a molten state. This also will result in the path of the wave packets of electrons being displaced away from the edges of the triangle of the lattice. This means that the "self-consistent potential" seen by an electron no longer has a component which has long-range order.

If the lattice melts into a compressible liquid, it is easily seen that all the "free energy" that is gained as a result of the large-ring exchanges is lost as a result of the phase incoherence arising from the long-wavelength density fluctuations. It can gain back part of that free energy, in spite of the disorder, by remaining as an incompressible fluid. Thus the phase factor in the action tries to lock the molten liquid into incompressible-liquid-like configurations. The two-body interaction favors triangular-lattice short-range order in this incompressible liquid.

It may be possible for the Wigner solid to melt even before the CRE sets in. This is because L becomes finite for $\alpha < \alpha_c/2$, where $\alpha_c \approx 1.1$. Thus the following is also an interesting possibility. As α is decreased, first the solid melts into a compressible liquid without CRE. Then it goes into an incompressible liquid having CRE.

We will now attempt to include liquidlike incompressible configurations as intermediate configurations in the treatment of KKAS. An incompressible configuration is defined by the following considerations. Any arbitrary configuration can be triangulated in a unique way.[9] We consider only those configurations whose mean number of triangles per unit area and mean area of the triangles a_0 coincides with that of a regular triangular lattice at the same density. Such configurations have well-developed short-range triangular-lattice order. Consider an arbitrarily large closed ring r of length $\sim l$ formed by the edges of the elementary triangles. Let the number of triangles enclosed by it be N_A. If the total area enclosed by this loop deviates from $N_A a_0$ (the mean area of N_A triangles) by an amount $\sim l^{1/2}$ or less, then we define this to be an incompressible configuration.

As mentioned before, such configurations have short-range triangular-lattice order and the deviation from long-range order arises through the occurrence of topological and nontopological defects in a triangular lattice. Simple considerations show that the nature of these defects and their spatial distribution are strongly constrained by the incompressibility requirement.

Associate an integer variable S_λ with every elementary triangle and let a_λ be its area. A ring r is formed by the edges of the triangles. The change in action due to a single ring-exchange event is

$$\alpha_r(\nu)\, n_r \pm 2\pi i\phi/\phi_0 + i\pi(n_r - 1), \qquad (2)$$

where n_r is the length of the ring r and ϕ/ϕ_0 is the total flux quanta enclosed by the ring. Since we have a random lattice, α depends on the ring r. We can write Eq. (2) as

$$\alpha_r(m)\, n_r \pm i\pi(m-1)N_A + 2i\pi m \textstyle\sum_\lambda (a_\lambda - a_0)/a_0 \pmod{2\pi}. \qquad (3)$$

The sum is over all triangles inside the ring r, a_0 is the area of the triangle of a periodic triangular lattice of the same density, and N_A is the number of triangles inside the ring r. Equation (3) follows from the fact that, as in the case of a triangular lattice, if the number of atoms in the ring is even (odd) then the number of triangles inside the ring is also even (odd) in a random triangulated lattice.

The action for a general exchange in terms of the spin variables is modeled as

$$H_{\mathrm{DG}} = \textstyle\sum \alpha_{\lambda\mu}(S_\lambda - S_\mu)^2 + ih(\nu)\sum(1+\epsilon\eta_\lambda)S_\lambda, \qquad (4)$$

where

$$\eta_\lambda \equiv (a_\lambda - a_0)/a_0, \quad \epsilon = 2m/(m-1). \qquad (5)$$

The phase factor $\pi(m-1)(1+\epsilon\eta_\lambda)$ is what every triangle λ will contribute to any ring exchange that encloses it. As in the case of KKAS theory, $\alpha_{\lambda\mu}$ is short ranged and positive. The dual lattice of the triangulated random network is not a simple honeycomb lattice. Its coordination, bond length, and bond angles vary as one moves along the lattice. Let C denote the topology or connectivity of a given random dual lattice. The connectivity C, by definition, uniquely specifies the coordination number of each vertex in the dual network. One can imagine a range of fluctuations in bond angles and bond lengths which keep the connectivity of the lattice the same. That is, consistent with the given connectivity the area of the elementary triangles a_λ can vary over a range.

We are going to consider paths which are in some sense "close" to the paths considered by KKAS. We assume that the melting instability discussed at the beginning of this Letter gives rise to a slow variation in the connectivity C of the intermediate configurations. This gives rise to a slow variation in a_λ. We also expect a fast variation in the configuration which does not change the connectivity drastically. The fast variation is analogous to the Gaussian fluctuation considered by KKAS around an exchange event.

First I will argue that in spite of the slow fluctuation in the area of the triangles we can have CRE as a result of the incompressibility constraint. In a compressible fluid the total area enclosed by the ring r is

$$A_r = \sum_{\lambda \in r} a_\lambda = N_A a_0 + \sum_{\lambda \in r} \eta_\lambda \tag{6}$$

$$\approx N_A a_0 \pm b_1 \sqrt{N_A} \pm b_2 \sqrt{l_r}, \tag{7}$$

where l_r is the perimeter of the ring and the b's are constants of the order of unity. The first term is N_r times the mean area of a triangle. The second term arises from the density fluctuation in the bulk which scales as the square root of the area of the loop. The last term arises from the boundary of the loop. For an incompressible fluid the area has the form

$$A_r \approx N_r a_0 \pm b_3 \pm b_2 \sqrt{l_r}. \tag{8}$$

The fluctuation from the bulk is replaced by a constant b_3 which is of the order of unity. The fluctuation for the incompressible liquid has exactly the same form as the rounding-off correction of KKAS which only adds a positive renormalization constant to α. Thus the phase incoherence arising from the slow variation in connectivity need not destroy CRE.

We can visualize the above in the following way. Large exchange ring paths are frozen in the liquid. In a compressible liquid these paths move adiabatically with the long-wavelength density fluctuations. This results in area fluctuation and hence the phase incoherence. In an incompressible fluid, however, the absence of long-wavelength density fluctuations suppresses large area fluctuations.

To study the fast fluctuations in the area, we assume that they are faster or of the order of the ring-exchange time scale. Then the partition function in the time slice τ_0 is

$$Z_{\tau_0}[C] = \sum_{\eta_\lambda^C} P[C, \eta_\lambda^C] Z[C, \eta_\lambda^C], \tag{9}$$

where $Z[C, \eta_\lambda^C]$ is the partition function for a given lattice configuration (C, η_λ^C) and $P[C, \eta_\lambda^C]$ is the probability that one will land at time t at a network with connectivity C and area of plaquettes $a_\lambda = a_0(1 + \eta_\lambda^C)$. Equation (9) resembles the annealed average of a random Gaussian model. The final partition function is approximately the product of $Z_{\tau_0}[C_i]$ over various time slices i.

The theory of KKAS amounts to assuming that the property of the product

$$P[C, \eta_\lambda^C] Z[C, \eta_\lambda^C] \tag{10}$$

is such that Eq. (9) is dominated by a C and $[\eta_\lambda^C]$ which corresponds to a perfect incompressible triangular lattice, including a certain amount of Gaussian fluctuations.

What is said in the present paper is that if as a result of the inherent melting instability Eq. (9) is dominated by a class of C and $[\eta_\lambda^C]$ which corresponds to an incompressible-fluid configuration, the CRE can still persist under some conditions. In order to consider the fast fluctuation we pick a typical disordered C and evaluate the partition function and average over η_λ^C. For convenience we will suppress the index C. First, let us assume that the η_λ are independent Gaussian random variables (this amounts to assuming that the fluid is compressible):

$$P[\eta_\lambda] = \frac{1}{(2\pi\sigma^2)^N} \exp\left(-\frac{1}{2\sigma^2} \sum \eta_\lambda^2\right), \tag{11}$$

where σ is a measure of the area fluctuation. The annealing is easily performed to get

$$Z_{\tau_0}[C] = \operatorname{Tr} \exp\left[-\sum \alpha_{\lambda\mu}(S_\lambda - S_\mu)^2 + ih \sum S_\lambda - \tfrac{1}{2}(h\sigma\epsilon)^2 \sum S_\lambda^2\right]. \tag{12}$$

The above annealed model has a "mass" term $\frac{1}{2}(\sigma\epsilon)^2 \sum S_\lambda^2$ and hence cannot have a roughening transition. This means that CRE is absent for any value of the constants $\alpha_{\lambda\mu}$. The mass term arises as a result of the incoherence arising from the density fluctuation in a compressible fluid. Even the introduction of short-range correlation in Eq. (11) does not remove the mass term.

The above suggests that if we want CRE we should introduce long-range correlation between the random variables $[\eta_\lambda]$. Since the Laughlin wave function describes an incompressible quantum fluid we can use an analytic form for correlation suggested by it. A change in area of an elementary triangle amounts to some local density fluctuation. Such density fluctuations get strong spatial correlation through the plasma correlation in the Laughlin wave function. This arises from the logarithmic interaction between the charges in the corresponding one-component plasma. Hence we assume a form

$$P[h_\lambda] \sim \exp\left(-\sum \eta_\lambda K_{\lambda\mu} \eta_\mu\right), \tag{13}$$

where

$$K_{\lambda\mu} \approx \ln|\mathbf{R}_\lambda - \mathbf{R}_\mu|, \text{ for large } |\mathbf{R}_\lambda - \mathbf{R}_\mu|. \tag{14}$$

The annealing can be performed with this distribution function to get

$$Z_{\tau_0} = \mathrm{Tr}\exp\left[-\sum \alpha_{\lambda\mu}(S_\lambda - S_\mu)^2 + ih\sum S_\lambda + (h\epsilon)^2 \sum K_{\lambda\mu}^{-1} S_\lambda S_\mu\right]. \tag{15}$$

Since $\sum_\mu K_{\lambda\mu}^{-1} = 0$ for every λ as a result of the asymptotic form of $K_{\lambda\mu}$,

$$\sum_{\lambda\mu} K_{\lambda\mu}^{-1} S_\lambda S_\mu \equiv \sum_{\lambda\mu} \alpha'_{\lambda\mu}(S_\lambda - S_\mu)^2. \tag{16}$$

Notice that the mass term is absent in the present annealed model only as a result of the logarithmic correlation in Eq. (13). Thus the effect of fluctuation in an incompressible fluid is only to renormalize the constants $\alpha_{\lambda\mu}$ to $\alpha_{\lambda\mu} + \alpha'_{\lambda\mu}$. It is intuitively obvious that the randomness in $\alpha + \alpha'$ is not very important as long as they do not change too often. For the presence of roughening and hence CRE the important thing is that one should have small enough $\alpha + \alpha'$ and the following symmetry in the Hamiltonian,

$$H[S_\lambda + n] = H[S_\lambda], \tag{17}$$

for any integer n.

The annealed Gaussian model can be transformed into the Coulomb gas by use of the transformation of Chui and Weeks.[10] In spite of the randomness in $\alpha + \alpha'$ the long-distance interaction in the dual model is of the Coulomb type. The randomness only affects the short-distance interaction between the charges. It is the long-distance behavior of the interaction which decides the presence of the dielectric phase (corresponding to the roughened phase in the discrete Gaussian model). Thus we have CRE. When we move away from the commensurate value by adding or subtracting extra particles, the nonanalytic change in energy depends only on the long-distance behavior of the interaction between the charges. Thus the cusp in the free energy and hence fractional quantization persists for sufficiently small values of $\alpha + \alpha'$. Notice that the incompressibility constraint also limits the "amount" of the short-time-scale fluctuation, leading only to a simple renormalization of α. Hence the configuration average over η done independently in each time slice does not really favor classical paths with large actions.

In this paper I have tried to show within the path-integral formalism of KKAS that incompressible-liquid-like configurations can sustain CRE under some conditions. I have also given arguments to suggest that the very existence of CRE in the present context implies a melting instability of the solid. This conclusion, together with the quantitative success of the Laughlin wave function and the observation that the Laughlin wave function contains CRE under some conditions, may strengthen the argument that CRE is the origin of the fractional quantum Hall effect.

It should be mentioned that recently Choquard and Clerouin[11] have suggested, on the basis of Monte Carlo studies, the cooperative ring motion to be the possible origin of the melting of the triangular Wigner solid of the two-dimensional one-component classical plasma. Also, Cross and Fisher[12] have suggested the possibility of CRE and raised the question of the stability of a solid in the presence of CRE in quantum solids.

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