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Dynamic Symmetries in Scattering

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We discuss the conditions that dynamic symmetries impose on the scattering S matrices and construct these matrices for problems with SO(2,1), SO(2,2), and SO(2,3) dynamical groups. The last group appears to be very useful for the analysis of heavy-ion collisions.

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Dynamical groups and spectrum-generating algebras have been used extensively in the description of bound states in a variety of problems in physics. The basic idea here is that the Hamiltonian (or mass operator) H belongs to the enveloping algebra of some group G, typically of the form

$$H=E_0+\sum_{\beta}\epsilon_\beta G_\beta+\tfrac{1}{2}\sum_{\beta\gamma}u_{\beta\gamma}G_\beta G_\gamma, \tag{1}$$

where the G_β's are generators of G, and E_0, ϵ_β, and $u_{\beta\gamma}$ are parameters which characterize the system under study. Dynamic symmetries occur whenever H can be written in terms only of Casimir invariants, C, of G (or of a string $G\supset G'\supset G''\supset\ldots$ of subgroups of G),

$$H=F(C). \tag{2}$$

Under these conditions, the eigenvalues of H can be obtained in closed form and provide energy or mass formulas of the type

$$E=F(\langle C\rangle), \tag{3}$$

where $\langle C\rangle$ denotes the expectation value of C in a representation of G. These formulas are very useful for comparison with experiments. Notable examples here are Gell-Mann–Ne'eman SU(3), which leads to the Gell-Mann–Okubo mass formula[1] and to the SU(6) generalization of Gürsey and Radicati[2] in elementary-particle physics; the dynamic symmetries of the interacting-boson model,[3] U(5), SU(3), and SO(6), in nuclear physics; the symmetry of the Coulomb potential,[4] SO(4), and its generalization to two-electron atoms[5] in atomic physics; and the symmetries of the vibron model,[6] U(3) and SO(4), in molecular physics.

Up to now, it has not been clear what the implications of dynamic symmetries and groups for scattering problems are, since the only known example of a dynamic symmetry in scattering has been that of the nonrelativistic scattering by a Coulomb potential [with symmetry group SO(3,1)]. We have found a way to construct S matrices starting from a dynamic group G. Application of this technique to several cases appears to indicate that the presence of a dynamic symmetry implies a particular functional form of the S matrix (and thus of the cross section), in the same way as the presence of a dynamic symmetry in bound-state problems implies a particular functional form of the eigenvalue spectrum. For example, we find that all problems with SO(3,1) symmetry imply S matrices which, for partial wave l and momentum k, have the form [SO(3,1)]

$$S_l(k)=\frac{\Gamma(l+1+if(k))}{\Gamma(l+1-if(k))}e^{i\phi(k)}. \tag{4}$$

The real function $f(k)$ is determined by the explicit expression of the Hamiltonian H in terms of Casimir invariants of G, while the phase $\phi(k)$ is determined by the associated asymptotic conditions. For example, in the Coulomb problem, we have

$$H=\alpha^2Z_1^2Z_2^2\mu c^2/2(C-1), \tag{5}$$

where C is the scalar quadratic invariant of SO(3,1) given in terms of the Lenz vector, $\hbar k\mathbf{A}/m$, and the angular momentum vector, $\mathbf{L}$, by $C=\mathbf{A}^2-\mathbf{L}^2$. Equation (5) and the asymptotic conditions on the Coulomb waves determine

$$f(k)=\alpha Z_1 Z_2 \mu c/\hbar k, \quad \phi(k)=0. \tag{6}$$

Here and in Eq. (5), α is the fine-structure constant, μ the reduced mass of the colliding particles, Z_1,Z_2 their charges, and $\hbar k$ their center-of-mass momentum.

The result stated above is important because it allows one to construct, in closed form, classes of solvable S matrices which can be used to analyze experimental data. However, there still remains the problem of how to go from a group G to the S matrix. By gaining experience from the simpler case[7,8] SU(1,1) $\approx$ SO(2,1), we have constructed, using purely algebraic methods, with no reference to the Schrödinger equation, S matrices for a class of problems of practical interest. These are of the type

$$\begin{aligned} &\mathrm{SO}(2,1)\approx \mathrm{SU}(1,1) \text{ for 1D},\\ &\mathrm{SO}(2,2)\approx \mathrm{SU}(1,1)\otimes \mathrm{SU}(1,1) \text{ for 2D},\\ &\mathrm{SO}(2,3) \text{ for 3D},\end{aligned} \tag{7}$$

and, in general, SO$(2,n)$ for problems in n space dimensions. The technique used is that of expansion of the eigenstates of SO$(2,n)$, which describe the scattering states in the presence of interactions, in terms of those of the corresponding Euclidean group, as in Ref. 8:

$$\begin{aligned} &\mathrm{SO}(2,1) \text{ expanded into } \mathrm{E}(2)\otimes \mathrm{E}(1),\\ &\mathrm{SO}(2,2) \text{ expanded into } \mathrm{E}(2)\otimes \mathrm{E}(2),\\ &\mathrm{SO}(2,3) \text{ expanded into } \mathrm{E}(2)\otimes \mathrm{E}(3),\end{aligned} \tag{8}$$

and, in general, SO$(2,n)$ expanded in terms of E(2) $\otimes$ E(n). In all these cases, the group E(2) describes the strength of the interaction, while the group E(n) describes the free waves in the n-dimensional space. Although not strictly necessary, we have also included explicitly in Eq. (8), in the one-dimensional case, the group E(1), in order to emphasize the general structure of the problem. Since the eigenstates of the Euclidean group E(n) describe the asymptotic incoming and outgoing free waves, the S matrix can be found from the expansion coefficients of SO$(2,n)$ into E(2) $\otimes$ E(n). Details of these expansions are contained in the thesis of one of us[9] and will be presented in a longer publication. Here we quote the results for the three-dimensional case, SO(2,3).

The eigenstates of SO(2,3) of interest in scattering problems are labeled by the quantum numbers $|\omega,l,m,v\rangle$, where

$$\begin{aligned} C_2|\omega,l,m,v\rangle &= \omega(\omega+3)|\omega,l,m,v\rangle,\\ \mathbf{L}^2|\omega,l,m,v\rangle &= l(l+1)|\omega,l,m,v\rangle,\\ L_3|\omega,l,m,v\rangle &= m|\omega,l,m,v\rangle,\\ V_3|\omega,l,m,v\rangle &= v|\omega,l,m,v\rangle.\end{aligned} \tag{9}$$

In Eq. (9), l and m denote the three-dimensional angular momentum and its z projection, v the strength of the interaction, and $\omega(\omega+3)$ the eigenvalues of the scalar quadratic Casimir invariant, C_2, of SO (2,3).

We note that, since we want to describe scattering states, we use in (9) the *continuous unitary representations* of SO(2,3), for which

$$\omega=-\tfrac{3}{2}+if(k), \tag{10}$$

where $f(k)$ is a real function of the center-of-mass momentum k. For such representations, the eigenvalue of the Casimir operator C_2 is $\frac{9}{4}-f^2(k)$, so that $f(k)$ is determined by the relation of the Hamiltonian to the Casimir invariant. Equation (10) is analogous to the relation $j=-\frac{1}{2}+if(k)$ for SO(2,1) discussed in Ref. 8.

The eigenstates of E(2) $\otimes$ E(3) can be written as $|\pm k,l,m,v\rangle$, where

$$\begin{aligned} \mathbf{P}^2|\pm k,l,m,v\rangle &= k^2|\pm k,l,m,v\rangle,\\ \mathbf{L}^2|\pm k,l,m,v\rangle &= l(l+1)|\pm k,l,m,v\rangle,\\ L_3|\pm k,l,m,v\rangle &= m|\pm k,l,m,v\rangle,\\ V_3|\pm k,l,m,v\rangle &= v|\pm k,l,m,v\rangle.\end{aligned} \tag{11}$$

In Eq. (11), k^2 denotes the eigenvalue of the Casimir invariant, $\mathbf{P}^2$, of E(3), i.e., the square of the momentum, while l,m, and v have the same meaning as before. [$\mathbf{P}^2$, $\mathbf{L}^2$, and L_3 are operators of E(3), while V_3 belongs to E(2).] Furthermore, $-k$ and $+k$ denote, respectively, the incoming- and outgoing-wave representations of the Euclidean group.

Expanding the asymptotic eigenstates, Eq. (9), in terms of Eq. (11),

$$|\omega,l,m,v\rangle = A_{lv}(k)|-k,l,m,v\rangle + B_{lv}(k)|+k,l,m,v\rangle, \tag{12}$$

and using the same algebraic technique discussed in Ref. 8, we find an S matrix

$$[S_l(k)=(-)^{l+1}B_{lv}(k)/A_{lv}(k)]$$

of the form

$$S_l(k)=\frac{\Gamma(\frac{1}{2}[l+v+\frac{3}{2}+if(k)])\Gamma(\frac{1}{2}[l-v+\frac{3}{2}+if(k)])}{\Gamma(\frac{1}{2}[l+v+\frac{3}{2}-if(k)])\Gamma(\frac{1}{2}[l-v+\frac{3}{2}-if(k)])}e^{i\Delta(k)}, \tag{13}$$

where $f(k)$ and $\Delta(K)$ are arbitrary real functions of k to be determined by the relationship between H and C and by the asymptotic conditions. If one requires that S_l approaches the Coulomb S matrix as $l \to \infty$, for any given k, then one must take in Eq. (13)

$$f(k) = \alpha Z_1 Z_2 \mu c/\hbar k, \quad \Delta(k) = 2\ln 2 f(k). \tag{14}$$

Pure Coulomb scattering can be obtained exactly from Eqs. (13) and (14) for any l and k by choosing $v = \frac{1}{2}$ in (13), as one can see by using the relation

$$\Gamma(z)\Gamma(z+\tfrac{1}{2}) = 2^{1-2z}\pi^{1/2}\Gamma(2z). \tag{15}$$

Cases with $v \neq \frac{1}{2}$ describe situations with a modified Coulomb interaction.

S matrices of the form (14) are thus very well suited to analysis of situations in which the underlying physical problem is that of a modified Coulomb interaction, as, for example, is the case in the scattering of two heavy ions. Here the interaction at large distances is the Coulomb interaction, while at short distances this interaction is modified by the nuclear contribution. Analysis of heavy-ion–reaction data directly in terms of S matrices will avoid the introduction of optical potentials, which may or may not exist, and which, in general, are nonlocal and energy dependent. In other words, the S matrix (14) may play the role here of the dual amplitude[10] used to analyze scattering data in elementary-particle physics in the early 1970's. Preliminary calculations appear to indicate that the S matrices (14) describe the data well. Results will be presented elsewhere.[11] We note here that, for v real, $S_l(k)$ satisfies all appropriate criteria for S matrices. For example, it is manifestly unitary. If one takes v complex, the unitary bound is satisfied if $\mathrm{Im}\, v^2 < 0$. This case describes scattering problems with absorption. Our approach can also be generalized to describe inelastic scattering, transfer reactions, and relativistic problems. Work in this direction is in progress.

In conclusion, it appears that spectrum-generating algebras and dynamic groups may also be of practical use in the analysis of scattering data.

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