# Linear and Nonlinear Theory of Cherenkov Maser Operation in the Intense Relativistic Beam Regime

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The linear dispersion relation for axisymmetric TM modes in a cylindrical waveguide lined with a dielectric material and enclosing a thin annular electron beam is derived and solved. Approximate analytic solutions are obtained for both weak and moderate beam regimes. A model of nonlinear saturation is developed. This model, combined with the linear theory in the moderate beam regime, defines the most efficient and compact Cherenkov maser.

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There have been a number of investigations of the so-called Cherenkov maser—an electron beam interacting with a dielectric-lined waveguide. Previous linear analysis confines itself to the "weak beam regime,"[1-8] solid beams,[2,3] plane geometry,[4] or the numerical solution of various dispersion relations that do not delineate the various aspects of the coupling. Nonlinear analyses and simulations[3,6,9,10] have limited the problem to interactions that are nonrelativistic in the wave frame. A number of different experiments have also been carried out.[7,8,11,12] Independently, high-brightness, intense, relativistic electron beams have been generated by use of a foilless diode.[13,14] Although developed for an inertial-confinement fusion concept that used the relativistic two-stream instability,[15] this class of relativistic electron beam might be applicable to high-power Cherenkov masers. Thus, we present a linear and nonlinear theory of the Cherenkov maser in a newly investigated regime associated with an annular, high-brightness, intense, relativistic electron beam. The analysis suggests that a high-power Cherenkov maser with 30% efficiency may be possible with this type of electron beam.

First, we obtain the linear dispersion relation for axisymmetric modes on a thin annular beam in a dielectric-line cylindrical waveguide. The present derviation is similar to previously published ones[1] and will only be outlined. Since axisymmetric TM modes decouple from TE modes, we need only consider the wave equation for the TM modes. This wave equation is solved in three regions, the first between the center of the waveguide and the annular beam, the second between the annular beam and the dielectric, and the third inside the dielectric. In general, each solution is expressed in terms of two independent eigenfunctions yielding a total of six constants to be determined by four boundary and two jump conditions. The boundary conditions are as follows: The longitudinal electric field $E_z$ is finite at the origin, $E_z$ vanishes at the conducting waveguide wall, and both $E_z$ and the radial electric displacement $\epsilon E_r$ are continuous across the vacuum-dielectric interface. Here $\epsilon$ is the dielectric constant. In the thin-beam approximation, the two jump conditions

$$[E_z]=0 \tag{1}$$

and

$$[rE_r]=2eikE_zI_b/(mv_b\gamma_b^3)(\omega-kv_b)^2 \tag{2}$$

are derived respectively from the Faraday and Gauss laws with a linearized cold-fluid description of the beam. In Eqs. (1) and (2) the square brackets indicate the enclosed quantity evaluated at the upper side of the beam at $r=r_b^+$ minus the quantity evaluated at the lower side of the beam at $r=r_b^-$. Furthermore $I_b$, $m$, $e$, $v_b$, and $\gamma_b$ are respectively the beam current, electron rest mass, electron charge, electron velocity, and Lorentz factor. For axisymmetric vacuum TM modes $E_r$ and $E_z$ are related by

$$E_r=(ik\,\partial E_z/\partial r)(\omega^2/c^2-k^2)^{-1}.$$

We have assumed a dependence of $\exp\{i(kz-\omega t)\}$ for the fields. The resulting dispersion relation is

$$(\pi/4)(\xi r_d)^2J_0(\xi r_d)[\Omega_b^2/(\omega-kv_b)^2][Q_2D_1-Q_1D_2]=[J_0(\xi r_d)D_1-J_1(\xi r_d)D_2], \tag{3}$$

where

$$D_1=\epsilon\xi[J_0(\rho r_w)Y_1(\rho r_d)-J_1(\rho r_d)Y_0(\rho r_w)],\quad D_2=\rho[J_0(\rho r_w)Y_0(\rho r_d)-J_0(\rho r_d)Y_0(\rho r_w)],$$

$$Q_1=J_1(\xi r_d)Y_0(\xi r_b)-J_0(\xi r_b)Y_1(\xi r_d),\qquad Q_2=J_0(\xi r_d)Y_0(\xi r_b)-J_0(\xi r_b)Y_0(\xi r_d),$$

$\xi^2=(\omega^2/c^2-k^2)$, $\rho^2=(\omega^2\epsilon/c^2-k^2)$, and $J_n$ and $Y_n$ are Bessel functions of the first kind and $n$th order. Also $\Omega_b^2=4I_bc^2/r_d^2\beta_b\gamma_b^3I_0$, where $I_0=e/mc^3$. Here $r_w$ stands for the conducting wall radius and $r_b$ for the beam radius. The dielectric extends from an inner radius $r_d$ to $r_w$. Equation (3) may also be recovered from Eq. (17) of Ref. 1

 

from Eq. (17) of Ref. 1 in the limit of a cold beam.

We have solved this dispersion relation numerically; we also seek a simple, approximate analytic solution appropriate to intense, relativistic beams. For this purpose, we make two approximations: (1) thin dielectrics, $(r_w - r_d)/r_w \ll 1$, and (2) small vacuum phase factors, $(\xi r_d)^2 \ll 4$. *A posteriori,* these conditions are equivalent to

$$1 \ll r_d/[2(r_w - r_d)] \ll \gamma_b^2(1 - 1/\epsilon) - 1. \qquad (4)$$

Taken together, these restrict the following analysis to thin but finite thickness dielectrics and relativistic beams. In this regime Eq. (3) reduces to a coupling between the dielectric-modified waveguide modes, $\omega = \pm\omega_k$, and two beam modes, $\omega = kv_b \pm \Omega_b$,

$$(\omega - \omega_k)(\omega + \omega_k)(\omega - kv_b + \Omega_b)(\omega - kv_b - \Omega_b) = 2\Omega_b^2 c^2/r_d(r_w - r_d), \qquad (5)$$

where $\omega_k = [k^2c^2/\epsilon + 2c^2/r_d(r_w - r_d)]^{1/2}$. Note that $\omega = \pm\omega_k$ are not the modes which arise by simple expansion around empty (no beam, no dielectric) waveguide modes.[7,8,11] In particular, the effective mode wave number perpendicular to the beam motion is not proportional to the waveguide radius $r_w$ but to the geometric mean of $r_w$ and the dielectric thickness $r_w - r_d$.

An instability arises whenever the forward waveguide mode, $\omega = \omega_k$, couples either to both beam modes when they are degenerate or to only the slow-beam mode. The first case we refer to as three-wave coupling and the second as to two-wave coupling. Here we are primarily concerned with beams strong enough to separate the slow- and fast-beam modes, and thus lead to two-wave coupling, but not strong enough to modify the waveguide modes. This "moderate beam regime" occurs for

$$\omega_i \ll 2\Omega_b \ll \omega_r, \qquad (6)$$

where $\omega_r$ and $\omega_i$ are the frequency and maximum growth rate of the corresponding instability, and, as we shall show, happens to be the regime of the most efficient, compact Cherenkov maser operation. Such an instability occurs at the resonance of the waveguide and slow-beam modes unmodified by the presence of current, respectively, $\omega = \omega_k$ and $\omega = kv_b$. Thus

$$\omega_r = [2c^2/r_w(r_w - r_d)]^{1/2}[1 - 1/\epsilon\beta_b^2]^{-1/2} \qquad (7)$$

and

$$\omega_i = [\Omega_b c^2/2r_w(r_w - r_d)\omega_r]^{1/2}. \qquad (8)$$

Other workers[1-8] have investigated only the corresponding "weak beam regime," $2\Omega_b \ll \omega_i$, in which $\omega_r$ is again given by Eq. (7), but $\omega_i = (\frac{3}{2})^{1/2} \times [\Omega_b^2 c^2/r_w(r_w - r_d)\omega_r]^{1/3}$. According to Eqs. (7) and (8), frequency is independent of $I_b$, growth rate in the moderate beam regime increases as $I_b^{1/4}$, and both are insensitive to thin-beam position $r_b$. In the strong beam regime, $\omega_r \ll 2\Omega_b$, $\omega_r$ is modified by beam current and the growth rate declines with $I_b$.

It has not been previously recognized that the nonlinear theory of a relativistic Cherenkov maser is similar to that of the relativistic two-stream instability. Thus, when the beam longitudinal velocity spread is small relative to the difference between the phase velocity of the most unstable wave and the mean beam velocity, the Cherenkov maser instability saturates by the trapping of beam electrons in the wave potential. Although beam energy spread and angular scatter both contribute to the longitudinal velocity spread, angular scatter is the dominant contribution in relativistic electron beams.[16]

Because foilless diodes require a strong magnetic field, a one-dimensional nonlinear analysis is appropriate. We assumed that the beam is initially homogeneous and that the trapping dynamics is determined by a single wave. Furthermore, the relative difference between the normalized initial beam velocity $\beta_b$ and the wave phase velocity $\beta_w$ is such that $\Delta = (1 - \beta_w/\beta_b) \ll 1$. Consider first the energy lost by an electron after being trapped and decelerated by the wave. Let $\gamma_b$ be the initial electron energy in the laboratory frame and $\gamma_f$ be the final electron energy in the laboratory frame after one-half rotation in phase space. Then the energy lost by a trapped electron is

$$(\gamma_f - \gamma_b)/\gamma_b = S/(1+S), \qquad (9)$$

where the strength parameter $S$ is given by

$$S = 2\beta_b^2\gamma_b^2\Delta. \qquad (10)$$

In the rigid-rotor model, beam electrons are trapped and decelerated together in the potential well. If this model were valid in the relativistic regime, the beam energy loss would be given by Eq. (9) and would approach unity for $S \gg 1$. But in the relativistic regime, trapping and deceleration lead to a spread in the rotation rate because of a spread in the effective mass of the electrons. However, it has been found that some fraction of the beam electrons do rotate in phase space coherently in a fashion similar to the rigid-rotor model.[17] The remaining electrons are spread out uniformly about the initial beam velocity, with a resulting redistribution of the electron distribution but little change in energy. The fraction of electrons that rotate coherently is $(\gamma_w/\gamma_b)^3$, where $\gamma_w = (1 - \beta_w^2)^{-1/2}$. One additional aspect of the trapping needs to be considered. As the electrons decelerate in the potential well, the wave becomes "loaded" and slows down. This leads to an effective 50% increase in the strength parameter.[16,18] It follows that the beam energy loss is

$$\Delta E/\gamma_b mc^2 = 1.5S(1 + 1.5S)^{-5/2}, \qquad (11)$$

where $S$, given by Eq. (10), is calculated from linear theory. In the weak beam limit, Eq. (11) predicts that the efficiency is proportional to $I_b^{1/3}$, consistent with simulation.[10] It is also correct for beams that are relativistic in the wave frame,

The energy loss predicted by Eq. (11) neglects beam scatter; this is accounted for in the so-called "quasihydrodynamic" model.[16] Unlike Eq. (11), which reduces to the nonrelativistic rigid-rotor result, the quasihydrodynamic model is not valid for nonrelativistic electrons beams. The model assumes that beam scatter further reduces the fraction of electrons that rotate coherently in phase space. With the angular scatter denoted by $\theta$, the quasihydrodynamic model gives

$$\Delta E/\gamma_b mc^2 = [1-\exp(-2\Delta/\theta^2)]1.5S(1+1.5S)^{-5/2}. \quad (12)$$

Thus, for the purpose of energy transfer, a beam is cold when $\theta^2 << 2\Delta$.

Note that the nonlinear evolution is determined by the wave phase velocity, not the growth rate. The instability growth rate and group velocity determine the interaction length. Thus, high efficiency and short interaction length are separate requirements. In particular, if the length of the dielectrics is not matched to the interaction length, the energy lost by the beam will be less than that given by Eq. (12). For a short system the instability has not saturated while for a long system the beam electrons absorb energy back from the wave.

A number of experiments have been reported. Here we describe the Cherenkov maser experiment with the highest power output.[12] A 2.8-mm-radius annular electron beam was propagated through a 80-cm-long dielectric cylinder with an inside radius of 3 cm and an outside radius of 5 cm. The dielectric was Texlite with a dielectric constant given by $\epsilon=3.67$. Peak power output was 580 MW. For a diode voltage of 650 keV the beam current was 6.5 kA. In evaluating this experiment, one must consider space-charge effects. If $\gamma_d$ is the diode voltage, the space-charge limiting current for an annular electron beam propagating through a dielectric-lined waveguide is

$$I_{\mathrm{sc}}(\mathrm{kA}) = \frac{8.5(\gamma_d^{2/3}-1)^{3/2}}{\ln(r_w/r_b)-(1-1/\epsilon)\ln(r_w/r_d)}. \quad (13)$$

Using Eq. (13) and energy conservation, we obtain a space-charge–depressed beam energy of 550 keV. The numerical solution of the dispersion relation, Eq. (3), predicts a wavelength of 3.6 cm and a strength parameter of 0.52 for this energy. A wavelength of 3.5 cm was observed. Equation (11) predicts an efficiency of 30% based on diode voltage, a factor of 2 higher than the 15% reported. However, in the experiment there appears to be a thin foil separating the diode region from the dielectric cylinder. No statement is made concerning the nature of this foil, but it could significantly reduce the efficiency by scattering the beam. For example, the quasihydrodynamic model, Eq. (12), reduces the efficiency to 20% for a 25-$\mu$m Ti foil. Also, the $e$-folding length for the instability is about 7.4 cm. Although it is difficult to estimate the number of $e$ folds required for saturation, as it depends upon the noise level on the beam, it is generally six to eight. But the dielectric length is nearly eleven $e$ folds, thus allowing wave energy to be transferred back into beam energy.

Significantly, given the beam quality and dielectric length, the parameters of this relatively successful experiment place it close to the most efficient one possible in the above described moderate beam or two-wave coupling regime. Efficient operation in the three-wave coupling regime is also possible, but convective growth lengths are longer than for the two-wave coupling regime because growth rates are smaller and phase velocities are larger. Therefore, the simultaneous requirements of efficiency and short interaction lengths dictate Cherenkov maser operation in the two-wave coupling regime.

A simple relationship among physical parameters describing this regime follows. According to Eq. (11), maximum energy loss occurs when $S=\frac{4}{9}$. With use of the definitions of $S$ and $\Delta$ and an approximate expression for $\beta_w$ appropriate to the two-wave coupling regime, $\Delta=\Omega_b/\omega_r$, where $\omega_r$ is given by Eq. (7), this fact becomes

$$\tfrac{2}{9}=\beta_b^2\gamma_b^2\Omega_b/\omega_r. \quad (14)$$

Given the definition of $\Omega_b$, this equation describes the relationship among beam current $I_b$ and other physical parameters which ensures the most efficient two-wave coupling. Now, using Eq. (14) to eliminate $\Omega_b$ from inequalities (4) and (6), we can describe this regime in terms of only the Lorentz factor $\gamma_b$, the dielectric constant $\epsilon$ and thickness $r_w-r_d$, and the wall radius $r_d$:

$$1 << r_d/(r_w-r_d) << \gamma_b^2 2(\epsilon-1)/\epsilon << \tfrac{82}{9}=9.12. \quad (15)$$

In summary, Eqs. (14) and (15), the requirement that beam scatter be small, and a proper choice of dielectric length define the most efficient, compact, Cherenkov masers possible.

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