Modular Invariance and One-Loop Finiteness of Five-Point Amplitudes in Type-II and Heterotic String Theories

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(Received 4 March 1986)

We prove the modular invariance and thus the finiteness of the five-point amplitudes in closed superstrings. The complicated algebra necessary is considerably simplified by making use of the results of Frampton, Moxhay, and Ng for the open superstring.

PACS numbers: 11.17+y

It is widely conjectured that superstring theories are finite, though a general proof does not yet exist. The basis of this hope is the classical demonstration of the finiteness of the one-loop four-point functions, for the SO(32) type-I superstring,[1] the type-II closed superstrings,[2] and the Spin(32)/Z_2 and $E_8\times E_8'$ heterotic strings.[3] A similar proof for the one-loop M-point functions, with $M>4$, is not straightforward because of the complexity of the algebra involved. This lengthy calculation has been accomplished recently for the $M=5$ type-I superstring[4,5] by Frampton, Moxhay, and Ng. There are two potential divergences in the type-I open string. The leading one is canceled between planar and nonplanar diagrams when the gauge group is SO(32), just like the case for $M=4$.[1] In addition, for $M=5$, there is also a potential nonleading divergence, which can be shown after a lengthy calculation to add up to zero.[5] Our purpose in this note is to prove the modular invariance, and thus the finiteness, of the $M=5$ type-II superstring as well as the Spin(32)/Z_2 and the $E_8\times E_8'$ heterotic strings. For simplicity we will restrict the external legs to be gauge bosons, which enables us to make use of the complicated calculations of Refs. 4 and 5.

First, consider a type-II closed superstring. The $M=5$ one-loop amplitude with external momenta k_i is given by[2,6]

$$A(1,2,3,4,5)=\left(\frac{\kappa}{4\pi}\right)^5\int\prod_{i=1}^{5}d^2z_i|w|^{-2}\left(\frac{-4\pi}{\ln|w|}\right)^5\prod_{1\le i<j\le 5}(\chi_{ij})^{k_i\cdot k_j/2}(I\cdot\tilde{I}+E), \tag{1}$$

where $w=z_1z_2z_3z_4z_5$, $\chi_{ij}=\chi(\nu_{ij},\tau)$,[6] $\nu_{ij}=\ln(z_{i+1}z_{i+2}\cdots z_j)/2\pi i$, and $\tau=\ln w/z\pi i$. In ν_{ij} and subsequently in (5), (6), and (29), we treat the indices i, j as cyclic, viz., $z_5\equiv z_0$, $z_6\equiv z_1$, $z_7\equiv z_2$, etc. Thus, if $\nu_i=\ln(z_1z_2\cdots z_i)/2\pi i$, then $\nu_{ij}=\nu_j-\nu_i$ if $j>i$, but $\nu_{ij}=\nu_j-\nu_i+\tau$ if $j<i$. The only property we need to know about $\chi(\nu,\tau)$ is its behavior under the modular transformation

$$\nu\to\nu'=-\nu/\tau,\quad \tau\to\tau'=-1/\tau. \tag{2}$$

It is[2]

$$\chi(\nu,\tau)=|\tau|\chi(\nu',\tau')=\chi(\nu+1,\tau)=\chi(\nu+\tau,\tau). \tag{3}$$

The function $I(\nu_i,\tau)=I_a+I_b+I_c+I_d$ can be taken from Refs. 4 and 5, with minor modifications. To study its behavior under (2) we need only to concentrate on its ν_i and τ dependences. I_a is a constant, independent of ν_i and τ, and

$$I_b=\sum_{1\le i<j\le 5}U_{ij}f_p(\nu_{ij},\tau), \tag{4}$$

$$I_c=-\sum_{i=1}^{5}\sum_{j=1}^{i-1}W_{ij}\mathrm{Im}(\nu_{ij})/\mathrm{Im}\tau, \tag{5}$$

$$I_d=\sum_{i=1}^{5}\sum_{j=1}^{i-1}W_{ij}f_p(\nu_{ij},\tau). \tag{6}$$

Here U_{ij} and W_{ij} are constant tensors, with

$$W_{ii}=0, \tag{7a}$$

$$\sum_{j=1}^{5}W_{ij}=0, \tag{7b}$$

which are consequences respectively of transversality of the polarization vector and the conservation of momentum. The expressions for I_a, U_{ij}, and W_{ij} are fairly complicated, but can be inferred from the expressions given in Refs. 4 and 5. Their detailed expressions will not be necessary for our purpose and will not be written down.

The planar mode sum function is[6]

$$f_p(\nu,\tau)=\sum_{l=1}^{\infty}\frac{1}{1-w^l}\left[z^l-\left(\frac{w}{z}\right)^l\right], \tag{8}$$

provided that $\nu=\ln z/2\pi i$ and $\tau=\ln w/2\pi i$. It obeys the relations

$$f_p(0,\tau)=1,\quad f_p(\nu-\tau,\tau)=-f_p(\nu,\tau). \tag{9}$$

Similarly, we can get the function $\tilde{I}_A$ by using $\tilde{I}_A(\nu_i,\tau)=I_A(\nu_i^*,\tau^*)$ $(A=a,b,c,d)$. The extra term E in (1) has the form

$$E=V/\mathrm{Im}\tau, \tag{10}$$

where V is a constant.

We give here a brief explanation of Eq. (1). Everything is worked out in Ref. 2, except for the factor $(I\cdot\tilde{I}+E)$. Let S_n and α_n ($\tilde{S}_n$ and $\tilde{\alpha}_n$) be the right-going (left-going) fermion and boson oscillator modes, and $R_0^{ij}=S_0^*\gamma^{ij-}S_0$ ($\tilde{R}_0^{ij}=\tilde{S}_0^*\gamma^{ij-}\tilde{S}_0$) be the S_0- ($\tilde{S}_0$-) dependent part of the vertex operators. For $M=4$, the term corresponding to $(I\cdot\tilde{I}+E)$ is proportional to the S_0, $\tilde{S}_0$ traces of $R_0^4\tilde{R}_0^4$. For $M=5$, there are four types of terms each[4,5] for the right-going and the left-going parts. They are (a) R_0^5, (b) $R_0^3(S_nS_0)^2$, (c) R_0^4p, and (d) $R_0^4\alpha_n (n\neq 0)$. When traces over S_0 are taken, and when corrections to the S_n traces (b), α_n traces (d), and internal loop momentum integration (c) are taken into account, we get the four terms I_A ($A=a-d$) in (4)–(6). Similarly we get $\tilde{I}_A$ when we consider the left-going modes. For right- (left-) going modes, the open-string variable x_i used in Refs. 4 and 5 is replaced by the complex variable z_i (z_i^*), which is how we obtained I_b and I_d ($\tilde{I}_b$ and $\tilde{I}_d$) in (4) and (6). Incidentally, (7) has been used to display I_c and I_d in the forms given by (5) and (6). Now for I_c and $\tilde{I}_c$. The internal momenta p_i for the ith propagator are common to the right and left modes; there is only one loop-momentum integration to be done, and not one each for right and left modes. If p is the loop momentum, then[2] $p_i=p-\sum_{j=1}^{i-1}k_j$ and the p integration is

$$J\equiv\int d^{10}p\prod_{i=1}^{5}|z_i|^{p_i^2/4}\equiv\int d^{10}p\exp[\alpha p^2-2\beta\cdot p+\gamma]=\left(\frac{-\pi}{\alpha}\right)^5\exp(\gamma-\beta^2/\alpha) \tag{11}$$

if no p is involved in the vertex operators. If p comes in linearly, then the corresponding integral with an extra p in the integrand of (11) is $(\beta/\alpha)J$. If p comes in quadratically, then the result is $[(\beta/\alpha)(\beta/\alpha)-1/2\alpha]J$. Note that the variable x_i of the open string is now replaced by $|z_i|$. This is why Eq. (5) involves $\mathrm{Im}(\nu)$ and $\mathrm{Im}(\tau)$, rather than ν, τ, or ν^*, τ^*. This is also why the left-handed modes and the right-handed modes do not quite factorize into $I\cdot\tilde{I}$; an extra term E in (1) corresponding to the term $(-1/2\alpha)J$ in the quadratic integral [coming from $(c)(\tilde{c})$] must be added. Since α in (11) is proportional to $\mathrm{Im}\tau$, E is proportional to $1/\mathrm{Im}\tau$, as indicated in Eq. (10).

Now we return to the proof of modular invariance of (1) under (2). The crucial observation, which we prove at the end, is that

$$I(\nu_i,\tau)=-\tau^{-1}I(\nu_i',\tau'). \tag{12}$$

Given this, and Eq. (10), we get

$$(I\cdot\tilde{I}+E)(\nu_i,\tau,\nu_i^*,\tau^*)=|\tau|^{-2}(I\cdot\tilde{I}+E)(\nu_i',\tau',\nu_i'^*,\tau'^*). \tag{13}$$

Moreover,

$$\left(\prod_{i=1}^{5}d^2z_i\right)\left(\frac{1}{\ln|w|}\right)^5=\left(d^2\tau\prod_{i=1}^{4}d^2\nu_i\right)\left(\frac{2\pi}{\mathrm{Im}\tau}\right)^5=\left(d^2\tau'\prod_{i=1}^{4}d^2\nu_i'\right)\left(\frac{2\pi}{\mathrm{Im}\tau'}\right)^5|\tau|^2. \tag{14}$$

Thus the additional powers of $|\tau|$ on the right-hand sides of (13) and (14) cancel each other. Since $\sum k_i\cdot k_j$, summed over $1\leqslant i<j\leqslant 5$, is zero for external massless states, the factor $|\tau|$ in (3) makes no contribution. This proves the invariance of (1) under (2).

For the heterotic string, we take the external particles to be charged gauge bosons. Then the five-point one-loop amplitude is proportional to[3,6]

$$A(1,2,3,4,5)=\int\prod_{i=1}^{5}d^2z_i|w|^{-2}\left(\frac{-4\pi}{\ln|w|}\right)^5\prod_{1\leqslant i<j\leqslant 5}(\chi_{ij})^{k_i\cdot k_j/2}I\tilde{T}, \tag{15}$$

with

$$\tilde{T}(\nu_i^*,\tau^*)=\left(\frac{1}{w^*}f(w^*)^{-24}\prod_{1\leqslant i<j\leqslant 5}[\psi(\nu_{ij}^*,\tau^*)^{K_i\cdot K_j}]\right)L(\nu_i^*,\tau^*). \tag{16}$$

Here k_i and K_i are the momenta in the ten-dimensional (external) and sixteen-dimensional (internal) spaces, respectively, for the ith charged gauge boson. The functions f and ψ emerge from the summation of the left-going (boson) oscillators, and L comes from the lattice sum over the sixteen-dimensional loop momenta. Under the modular transformation (2), these functions transform as follows[2,3]:

$$\psi(\nu^*,\tau^*)=-\tau^*\exp(-\pi i\nu^{*2})\psi(\nu'^*,\tau'^*), \tag{17}$$

$$w^*f(w^*)^{24}=w'^*f(w'^*)^{24}/\tau^{*12}, \tag{18}$$

$$L(\nu_i^*,\tau^*)=\exp\left[-\frac{\pi i}{\tau^*}\left(\sum_{j=1}^{5}\mu_j^*Q_j\right)^2\right]L(\nu_i'^*,\tau'^*)/\tau^{*8}=\exp\left[+\frac{\pi i}{\tau}\sum_{1\leqslant i<j\leqslant 5}\nu_{ij}^{*2}K_i\cdot K_j\right]L(\nu_i'^*,\tau'^*)/\nu_{ij}^{*2}. \tag{19}$$

Here $Q_i = \sum K_j$ (summed over $1 \leq j \leq i-1$), $Q_1 = 0$, and $\mu_j^* = \ln z_j^*/(-2\pi i)$. In getting to the last form of (19), momentum conservation $\sum K_i = 0$ has been used. Since $\sum K_i \cdot K_j$, summed over $1 \leq i < j \leq 5$, is $\sum K_i^2/2 = -5$, we get from (16)–(19) that

$$\tilde{T}(\nu_i^*, \tau^*) = -\tau^{*-1}\tilde{T}(\nu_i'^*, \tau'^*). \tag{20}$$

In other words, $\tilde{T}$ transforms just like $\tilde{I}$ [see (12)]. Thus Eq. (15) for the heterotic string is also modular invariant.

We proceed now to prove the crucial relation (12). To do this, we make use of the result in Refs. 4 and 5, which states that the ν-independent terms of $I_a + I_b$, as well as the ν-linear terms of $I_a + I_b$, add up to zero. This result is used there to show the absence of the nonleading divergence of the SO(32) open superstring. In practice, ν independent and ν linear correspond to replacement of $f_p(\nu,\tau)$ in I_b by $-\frac{1}{2}$ and ν, respectively, and hence this result can be stated as follows:

$$I_a - \frac{1}{2}\sum_{1 \leq i < j \leq 5} U_{ij} = 0, \tag{21}$$

$$\sum_{1 \leq i < j \leq 5} U_{ij}\nu_{ij} = 0. \tag{22}$$

From (4), using (21) and (22), we may write

$$I_a + I_b = \sum_{1 \leq i < j \leq 5} U_{ij}\hat{f}_p(\nu_{ij}, \tau), \tag{23}$$

where

$$\hat{f}_p(\nu,\tau) = f_p(\nu,\tau) + \tfrac{1}{2}(1 - \nu/\tau). \tag{24}$$

Because of (22), the coefficient of the last term in (24) is not unique. That particular choice, however, leads to a simple transformation law for $\hat{f}_p$,

$$\hat{f}_p(\nu,\tau) = -\hat{f}_p(\nu',\tau')/\tau, \tag{25}$$

as we shall now proceed to prove. In terms of the Jacobi θ function θ_3,[7] we can write (8) as

$$f_p(\nu,\tau) = \frac{i}{2\pi}\frac{\theta_3'(\nu - \frac{1}{2}\tau - \frac{1}{2}|\tau)}{\theta_3(\nu - \frac{1}{2}\tau - \frac{1}{2}|\tau)}. \tag{26}$$

Since

$$\theta_3(\nu|\tau) = (-i\tau)^{-1/2}\exp(-\pi i\nu^2/\tau)\theta_3(\nu'/\tau'), \tag{27}$$

we get

$$f_p(\nu,\tau) = \tau' f_p(\nu',\tau') + \tfrac{1}{2}\tau' - \nu' - \tfrac{1}{2}, \tag{28}$$

from which (25) follows.

Because of (25), $I_a + I_b$ has the desired transformation property of (12). For $I_c + I_d$, (5), (6), (7b), and (24) give

$$I_c + I_d = \sum_{i=1}^{5}\sum_{j=1}^{i-1} W_{ij}\left[\hat{f}_p(\nu_{ij},\tau) + \frac{1}{2}\frac{\nu_{ij}}{\tau} - \frac{\mathrm{Im}(\nu_{ij})}{\mathrm{Im}(\tau)}\right]. \tag{29}$$

Now

$$\frac{\mathrm{Im}(\nu')}{\mathrm{Im}(\tau')} = \frac{\mathrm{Im}(\nu/\tau)}{\mathrm{Im}(1/\tau)} = \frac{\mathrm{Im}(\nu)\mathrm{Re}(\tau) - \mathrm{Re}(\nu)\mathrm{Im}(\tau)}{-\mathrm{Im}(\tau)} = -\frac{\mathrm{Im}(\nu)}{\mathrm{Im}(\tau)}\tau + \nu,$$

and thus

$$\frac{\mathrm{Im}(\nu')}{\mathrm{Im}(\tau')} - \frac{1}{2}\frac{\nu'}{\tau'} = (-\tau)\left[\frac{\mathrm{Im}(\nu)}{\mathrm{Im}(\tau)} - \frac{1}{2}\frac{\nu}{\tau}\right]. \tag{30}$$

Using (25) and (30), we see that $I_c + I_d$ in (29) also has the transformation property of (12).

This completes the proof of (12). For full modular invariance, we still need to show the invariance of the integrands of (1) and (15) under (A) $\nu_k \to \nu_k + 1$ $(1 \leq k \leq 4)$, (B) $\nu_k \to \nu_k + \tau$, and (C) $\tau \to \tau + 1$. (C) is trivial as $w = \exp(2\pi i\tau)$ remains unchanged. Under the transformation (A), $z \equiv \exp(2\pi i\nu_{ij})$, and hence by (8) $f_p(\nu_{ij},\tau)$ are unchanged; so is $\mathrm{Im}\nu_{ij}$, and $\chi(\nu_{ij},\tau)$. Thus the integrand of (1) is invariant. The integrand of (15) will also be invariant if $\tilde{T}$ is, and the proof of that is identical to Eq. (6.17) in the second paper of Ref. 3. Now consider (B). This implies $\nu_{ij} \to \nu_{ij} + \tau(\delta_{jk} - \delta_{ik})$. By (26) and the known transformation property of θ_3,[7] $f_p(\nu_{ij},\tau) \to f_p(\nu_{ij},\tau) + \delta_{jk} - \delta_{ik}$, and thus by (5) and (6) the net change of $I_c + I_d$ is zero. The function I_b in (4) now changes into $I_b + X_k$, where X_k is the sum of U_{ik} for all $i < k$ minus the sum of all U_{kj} for all $j > k$. But (22) can be true for arbitrary ν_i only when all such quantities X_k vanish. Thus I_b and hence I are invariant under (B). Combining this with (3), we see that Eq. (1) is invariant. From Ref. 3 we know that $\tilde{T}$, and hence (15), is also invariant. This then completes the proof of the modular invariance of the type-II superstring and the heterotic string. This then allows us to integrate over a fundamental region away from $|\tau| = 0$,[2,3] thereby avoiding all possible divergences.

This research is supported in part by the Natural Science and Engineering Research Council of Canada and the Québec Department of Education.

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