Electrically Charged Vortices in Non-Abelian Gauge Theories with Chern-Simons Term

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It is shown that a non-Abelian gauge theory with Higgs fields and the addition of a Chern-Simons term in $2+1$ space-time dimensions exhibits finite-energy electrically charged vortex solutions. A novel feature of the vortices is that their electric charge Q is quantized in units of the fundamental charge e, $Q/e=n/2$ with n an integer, and their angular momentum is $J=Q/2e=n/4$.

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Gauge theories exhibit a rich spectrum of finite-energy (or finite-action) classical solutions. Vortices, monopoles, and instantons are the best known topological solutions in 2, 3, and 4 space dimensions, respectively.[1] Magnetic monopoles admit an electrically charged generalization in $3+1$ space-time dimensions: dyons. Vortices, in both Abelian[2] and non-Abelian Higgs models,[3-5] do not admit finite-energy charged generalizations in $2+1$ dimensions.[6,7]

The addition of a Chern-Simons (CS) term[8-10] to the Abelian Higgs model (in $2+1$ dimensions) changes this situation.[11] In fact, as a result of the presence of this term, vortices acquire electric charge keeping finite energy. An Abelian vortex with k units of magnetic flux has an electric charge $(2\pi\mu/e)k$, where μ, which has the dimension of a mass, is the coefficient of the CS term and e is the fundamental charge unit.

We present in this note finite-energy charged solutions for a non-Abelian Higgs model with a CS term in $2+1$ dimensions. In this case, the CS term is topological in character and noninvariant under "large" gauge transformations, this last implying that μ must be quantized.[10] As we shall see, this results in the quantization of the vortex electric charge Q which turns to be $Q=(e/2)n$ with n an integer; as a consequence, the associated angular momentum of the vortex is $J=n/4$.

Vortex solutions are associated with spontaneously broken gauge symmetries via Higgs fields. In order to have topologically stable vortices, the relevant homotopy group, $\pi_1(G/H)$, must be nontrivial. (G stands for the gauge group and H for the invariance group of the vacuum.) For $G=\mathrm{SU}(N)$ and the Higgs fields in the adjoint representation it is convenient to have maximum symmetry breaking of G so that the vacuum is only invariant under the unit matrix in the adjoint representation. Then $H=Z_N$, $\pi_1(\mathrm{SU}(N)/Z_N)=Z_N$, and we have $N-1$ topologically nontrivial homotopy classes besides the ordinary vacuum (trivial class).

Although the arguments that follow are general and apply to "realistic" theories such as the SU(5) grand unified theory, it is useful to consider the simplest theory that admits non-Abelian vortices.[3-5] This is the theory of a gauge field A_μ taking values in the Lie algebra of SU(2) (with generators t^a) coupled to two Higgs fields (in order to achieve maximum symmetry breaking) in the adjoint representation. The Lagrangean density (in $2+1$ space-time dimensions) reads

$$\mathscr{L}=-\tfrac{1}{4}\mathbf{F}_{\mu\nu}\cdot\mathbf{F}^{\mu\nu}+\tfrac{1}{2}D_\mu\boldsymbol{\phi}\cdot D^\mu\boldsymbol{\phi}+\tfrac{1}{2}D_\mu\boldsymbol{\psi}\cdot D^\mu\boldsymbol{\psi}+\tfrac{1}{4}\mu\epsilon_{\mu\nu\alpha}[\mathbf{F}^{\alpha\mu}\cdot\mathbf{A}^\nu-\tfrac{2}{3}e\mathbf{A}^\alpha\cdot(\mathbf{A}^\mu\times\mathbf{A}^\nu)]-V(\boldsymbol{\phi},\boldsymbol{\psi}) \tag{1}$$

with $A_\mu=A_\mu^a t^a$ and

$$D_\mu\boldsymbol{\phi}=\partial_\mu\boldsymbol{\phi}+e\mathbf{A}_\mu\times\boldsymbol{\phi},\quad \mathbf{F}_{\mu\nu}=\partial_\mu\mathbf{A}_\nu-\partial_\nu\mathbf{A}_\mu+e\mathbf{A}_\mu\times\mathbf{A}_\nu.$$

The fourth term on the right-hand side of Eq. (1) is the Chern-Simons term. It violates both P and T invariance but not C invariance. Although it leads to gauge-covariant equations of motion, it is not itself gauge invariant; rather, it changes by a total derivative. It then follows that the response of the action to a gauge transformation u is

$$S=\int d^3x\,\mathscr{L}\to\int d^3x\,\mathscr{L}+\mu(8\pi^2/e^2)\omega(u),$$

where $\omega(u)$ is the winding number of the gauge transformation. In order to make $\exp(iS)$ gauge invariant, one then has to impose

$$\mu=-e^2n/4\pi,\quad n\in Z. \tag{2}$$

For $G=\mathrm{SU}(2)$ there is only one topologically stable vortex with $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ not parallel in isospace at the minimum of the potential $V(\boldsymbol{\phi},\boldsymbol{\psi})$ which can be taken, for example, as

$$V(\boldsymbol{\phi},\boldsymbol{\psi})=\tfrac{1}{8}g^2(\boldsymbol{\phi}^2-\eta^2)^2+\tfrac{1}{8}g'^2(\boldsymbol{\psi}^2-\eta'^2)^2+\tfrac{1}{2}g''^2(\boldsymbol{\psi}\cdot\boldsymbol{\phi})^2. \tag{3}$$

We make the following *Ansatz* for the charged vortex solution:

$$\boldsymbol{\phi}=f(\rho)(\cos\varphi,\sin\varphi,0),\quad \boldsymbol{\psi}=\hat{\mathbf{e}}_3\eta',$$
$$\mathbf{A}_\varphi=\hat{\mathbf{e}}_3A(\rho),\quad \mathbf{A}_0=\hat{\mathbf{e}}_3A_0(\rho) \tag{4}$$

with $\hat{\mathbf{e}}_3=(0,0,1)$ and (ρ,φ) polar coordinates. One can also make another vortex *Ansatz* exchanging the coordinate dependence of $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$. It can be seen that Eq. (4) leads to the lowest vortex solution provided $\eta<\eta'$ and $g<g'$ or $m<m'$ and $g>g'$ (where m is the Higgs mass, $m=\eta g$).

Finite action requires the following boundary conditions:

$$A(\infty)=-1/e,\quad A_0(\infty)=0,\quad f(\infty)=\eta,$$

and

$$A(0)=A_0(0)=f(0)=0.$$

In order to define an electromagnetic field $\mathscr{F}_{\mu\nu}$ we have at hand two Higgs fields. However, our *Ansatz* corresponds to $\mathbf{F}_{\mu\nu}$ and $\boldsymbol{\phi}$ mutually orthogonal. We hence propose

$$\mathscr{F}_{\mu\nu}=(\boldsymbol{\psi}/|\boldsymbol{\psi}|)\cdot\mathbf{F}_{\mu\nu}=F^3_{\mu\nu}. \tag{5}$$

The electric and magnetic fields take then the form

$$E_i=\mathscr{F}_{0i},\quad B=\tfrac{1}{2}\epsilon_{ij}\mathscr{F}^{ij}. \tag{6}$$

The topological charge k of the vortex configuration is related to the magnetic flux: $k=e\Phi/2\pi$ with $\Phi=\int d^2x\,B$. From the Stokes theorem and the boundary conditions one gets

$$\Phi=-2\pi/e \tag{7}$$

and hence $k=-1$ for our solution. With *Ansatz* (4) the equations of motion

$$D_\mu D^\mu\boldsymbol{\phi}=-\frac{\delta V}{\delta\boldsymbol{\phi}},\quad D_\mu D^\mu\boldsymbol{\psi}=-\frac{\delta V}{\delta\boldsymbol{\psi}},$$
$$D_\mu\mathbf{F}^{\mu\nu}=\mathbf{j}^\nu+\tfrac{1}{2}\mu\epsilon^{\nu\alpha\beta}\mathbf{F}_{\alpha\beta}, \tag{8}$$
$$\mathbf{j}^\nu\equiv e(D^\nu\boldsymbol{\phi}\times\boldsymbol{\phi}+D^\nu\boldsymbol{\psi}\times\boldsymbol{\psi})$$

become

$$A_0''(\rho)+\rho^{-1}A_0'(\rho)-e^2f^2A_0(\rho)=-(\mu/\rho)A'(\rho),\quad A''(\rho)-\rho^{-1}A'(\rho)-ef^2(1+eA)=-\mu\rho A_0'(\rho),$$
$$f''(\rho)+\rho^{-1}f'(\rho)-\rho^{-2}(1+eA)^2/f(\rho)+e^2A_0^2f=\tfrac{1}{2}g^2(f^2-\eta^2)f. \tag{9}$$

As a result of the presence of the CS term only charged solutions are possible since Eq. (9) leads to a trivial $\mathbf{A}$ for $A_0=0$. One can easily analyze the asymptotic behavior of the charged solution from Eqs. (9). One finds two possible solutions for large ρ:

$$A(\rho)=-1/e+Z_\pm(m\rho/e)K_1(m_\pm\rho)[1+O(e^{-m_\pm\rho})],\quad A_0(\rho)=\pm Z_\pm(m/e)K_0(m_\pm\rho)[1+O(e^{-m_\pm\rho})],$$
$$f(\rho)=\eta[1-Y_\pm K_0(m\rho)+O(e^{-2m\rho})], \tag{10}$$

where $Z_\pm$, $Y_\pm$ are dimensionless constants and $m_\pm$ the two distinct vector meson masses,

$$m_\pm=(\tfrac{1}{4}\mu^2+e^2\eta^2)^{1/2}\pm\tfrac{1}{2}\mu. \tag{11}$$

Concerning the electric charge of the vortex configuration, note that the first of Eqs. (9) can be written in the form

$$\partial^iE_i+\mu B=\sigma, \tag{12}$$

where the charge density σ reads

$$\sigma\equiv\hat{\mathbf{e}}_3\cdot\mathbf{J}_0=e^2f^2A_0.$$

Since $\lim_{\rho\to\infty}E_i=0$ one gets from Eq. (12) a relation between the electric charge Q,

$$Q=\int d^2x\,\sigma, \tag{13}$$

and the magnetic flux:

$$Q=\mu\Phi. \tag{14}$$

This important relation, already recognized in Ref. 10, is also valid in the Abelian model. In the present non-Abelian case, it implies a quantization condition for Q. Indeed, from Eqs. (2), (7), and (14) we get

$$Q=(e/2)n,\quad n\in Z. \tag{15}$$

The charge quantization can be connected with the angular momentum of the vortex. In two space dimensions there is only one generator of angular momentum:

$$J=\int d^2x\,\epsilon^{ij}x_iT_{0j},$$

where $T_{\mu\nu}$ is the energy-momentum tensor. For the vortex solution, the only nontrivial momentum density is $T_{0\varphi}$. Using its explicit form one gets

$$J=2\pi\int_0^\infty\rho\,d\rho\,eA_0f^2+\tfrac{1}{2}\mu\pi A(\infty)^2$$
$$=Q/e+\pi\mu/e^2$$

or

$$J=Q/2e.$$

Hence, the vortex carries a quantized angular momentum, $J=n/4$ (in contrast with the Abelian case where, μ not being quantized, it can take any continuum value).

As is well known, vortices exist provided a Ginzburg-Landau–type parameter λ, which in the Higgs model is related to the scalar and vector meson masses,

$$\lambda = m/m_{\text{vector}}$$

satisfies the condition $\lambda > 1$ corresponding to type-II superconductivity. This leads to

$$\mu = ne^2/4\pi > g\eta(e^2/g^2 - 1)$$

for $m_{\text{vector}} = m_+$ and

$$0 < -ne^2/4\pi < g\eta(e^2/g^2 - 1)$$

for the m_- solution (this is also valid in the Abelian model[11]). Concerning the energy ϵ of the vortex solution, from the asymptotic fields, Eq. (10), it follows that $\epsilon \sim m_\pm^2 \ln(e^2/m_\pm^2)$ and hence the m_- solution has lower energy.

Qualitatively, the field behavior is the following: The magnetic field decreases monotonically from its value at the origin to zero at infinity with characteristic length $1/m_-$; the scalar field ϕ increases with characteristic length $1/m$ from zero at the origin to its vacuum value at infinity. Finally, the electric field vanishes both at the origin and at infinity, reaching its maximum at some finite ρ.

As we stated above, an SU(N) theory admits vortices with topological charge $k = 1, 2, \ldots, N-1$. In that case, the charge $Q = (en/2\sqrt{N})a$ with $a = 1, 2, \ldots, N-1$.

We conclude by noting that (Euclidean) three-dimensional field theories can be used to study four-dimensional physics at high temperatures. In this context, charged vortex lines can find important applications in the early universe.

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