# Exact Integrability of the One-Dimensional Hubbard Model

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The 1D Hubbard model is shown to be an exactly integrable system. A "covering" model of 2D statistical mechanics which I proposed recently was shown to provide a one-parameter family of transfer matrices, commuting with the Hamiltonian of the Hubbard model. I show in this work that any two transfer matrices of a family commute mutually. At the root of the commutation relation is the ubiquitous Yang-Baxter factorization condition. The form of the $R$ operator is displayed explicitly.

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The 1D Hubbard model is of considerable interest in solid-state physics. It is exactly solvable by the Bethe *Ansatz*.[1] In this work I establish its exact integrability as well. Exact integrability, as is well known, is a very powerful result and is encountered in several classic models in statistical mechanics, such as the 2D Ising, the *XYZ*, and the eight-vertex models. A commonly accepted feature of exactly integrable models is the existence of an infinite number of conserved "currents," and of their mutual commutation. This is usually established, for a quantum Hamiltonian, by the identification of a "covering" lattice-statistical model with the property that a one-parameter family of transfer matrices commutes with the Hamiltonian. The mutual commutation of two transfer matrices (of the same family) establishes the exact integrability.

In a previous work,[2] I identified a new model in 2D classical statistical mechanics, and showed that it is a "covering" model for the 1D Hubbard model (in the above sense). In this work, I prove that two transfer matrices of the same family commute mutually. This is shown by a demonstration that the ubiquitous star-triangle[3,4] (Yang-Baxter) relation holds in this case as well.

The model considered here consists of two six-vertex models, obeying the free Fermi condition and coupled in a particular way by a diagonal vertex. The transfer matrix is written as $T=\mathrm{tr}_g Y$, with

$$\begin{aligned} &Y=L_{N,g}L_{N-1,g}\cdots L_{1,g},\quad L_{n,m}=I_m l_{n,m} I_m, \text{ with } l_{n,m}=S_{n,m}T_{n,m},\\ &S_{n,m}=\tfrac{1}{2}(a+b)+\tfrac{1}{2}(a-b)\sigma_n^z\sigma_m^z+c(\sigma_n^+\sigma_m^-+\text{H.c.}), \end{aligned} \tag{1}$$

and $I_m=\exp(\frac{1}{2}h\sigma_m^z\tau_m^z)$. $T_{n,m}$ has the same form as $S_{n,m}$ with $\sigma$'s replacing $\tau$'s.

The model thus consists of two species of Pauli matrices $\sigma$ and $\tau$ residing at sites $n=1,\ldots,N$, and periodic boundary conditions are assumed. The spin at site $g$ is an auxiliary variable (the ghost spin) corresponding to a horizontal arrow, and Y is the monodromy matrix. The model is characterized by three distinct parameters $a/c$, $b/c$, and $h$, with $a^2+b^2=c^2$ (we set $c=1$ in the following). In Ref. 2 I showed that $T$ commutes with the Hubbard Hamiltonian $H$ provided that

$$(2/ab)\sinh(2h)=U, \tag{2}$$

and further that $H$ is a logarithmic derivative of $T$. (Here $U$ is the Coulomb constant in $H$). Let us note that $L_{n,m}$ is asymmetric in $n$ and $m$.

Next let us consider two transfer matrices ($T$ and $T'$) with parameters $a_1,b_1,h_1$ and $a_2,b_2,h_2$, both obeying the free Fermi condition, and examine the commutator. Clearly

$$\begin{aligned} YY'&=(L_{N,g_1}L'_{N,g_2})\cdots(L_{1,g_1}L'_{1,g_2}),\\ Y'Y&=(L'_{N,g_1}L_{N,g_2'})\cdots(L'_{1,g_1}L_{1,g_2'}). \end{aligned}$$

Taking the trace over $g_1$ and $g_2$ of the two equations and subtracting, we obtain the commutator $[T,T']$. Baxter[3] noted that the commutator vanishes provided that the two expressions are similarity transforms of one another, and found a *local* relation that is sufficient. This may be written in the form

$$L_{3,2}L'_{3,1}R_{12}=R_{12}L'_{3,2}L_{3,1} \tag{3}$$

(with $g_1\to 2$, $g_2\to 1$, and $n\to 3$). The operator $R_{12}$ does not depend on the site 3 (or $n$), and provided that it is invertible, the commutator vanishes. The remainder of this paper is concerned with the demonstration that Eq. (3) holds in the present model, and with the determination of $R_{12}$.

As a prelude to the calculation let us first consider the infinitesimal case $\theta_2=\epsilon$ ($a_2=\cos\theta_2$, $b_2=\sin\theta_2$, $c_2=1$), to first order in $\epsilon$, and $h_2$ determined from Eq. (2). This gives $L'_{3,1}=P_{31}[1+\epsilon H_{3,1}]$ where $P_{31}$ is the permutation operator and

$$\begin{aligned} H_{3,1}=&(\sigma_3^+\sigma_1^-+\text{H.c.})\\ &+(\sigma\to\tau)+\tfrac{1}{8}U(\sigma_3^z\tau_3^z+\sigma_1^z\tau_1^z). \end{aligned}$$

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Thus to order $\epsilon$, $T' = T(0)[1+\epsilon H + O(\epsilon^2)]$, where $H$ is the Hubbard Hamiltonian $\sum H_{n,n+1}$. We know already (from Ref. 2) that $T$ must commute with $T'$ to this order, and examine if this commutation relation can be used to extract $R_{12}$ [to $O(\epsilon)$]. Writing $R_{12} = P_{12}V_{12}$ and $V_{12} = U_{12} + \epsilon W_{12} + O(\epsilon^2)$ we find from Eq. (3) two equations to order $\epsilon$: $L_{21}U_{12} = U_{13}L_{31} \Rightarrow U_{12} = L_{21}^{-1}$, and

$$L_{21}H_{32}L_{21}^{-1} - L_{31}^{-1}H_{32}L_{31} = W_{13}L_{31} - L_{21}W_{12}. \quad (4)$$

The left-hand side of Eq. (4) is also encountered in Ref. 2, where it is noted to be $M_{2,1}^{\dagger} - M_{3,1}$, with $M$ a non-Hermitean operator [Eq. (8) of Ref. 2]. This remarkable separation of variables requires only the free Fermi condition and may be used to write the general solution for $W$,

$$L_{21}W_{12} = -M_{2,1}^{\dagger} + f_1, \quad W_{13}L_{31} = -M_{3,1} + f_1,$$

where $f_1$ is an arbitrary operator depending on site 1 only. Writing $M_{3,1} = L_{3,1}^{-1}Q_{3,1}$, we have

$$W_{12} = -L_{21}^{-1}Q_{2,1}^{\dagger}L_{2,1}^{-1} + L_{2,1}^{-1}f_1,$$
$$W_{13} = -L_{31}^{-1}Q_{3,1}L_{3,1}^{-1} + f_1L_{3,1}^{-1}. \quad (5)$$

Demanding the equality of the two expressions $(2 \to 3)$ we find a constraint

$$L_{3,1}^{-1}(B_{3,1})L_{3,1}^{-1} = \tfrac{1}{2}[L_{3,1}^{-1}, f_1] \quad (6)$$

with $B = (Q^{\dagger} - Q)/2$. However, I showed in Ref. 2 that provided that Eq. (2) holds, $B_{3,1} = \alpha[L_{3,1}I_1^{-4}]$, where $\alpha = (c^2+2b^2)/4ab$. Therefore, $f_1 = -2\alpha I_1^{-4}$, and $R_{12}$ exists and can be found to $O(\epsilon)$.

We now turn to the general problem, Eq. (3), and seek to determine $R$. The symmetries of the model lead us to require the commutation of $R_{12}$ with the operators (symmetry generators) (1) $\sigma_1^x\sigma_2^x\tau_1^x\tau_2^x$ (conjugation of all arrows), (2) $\sigma_1^z + \sigma_2^z$ (conservation of particles of $\sigma$ species), and (3) $\tau_1^z + \tau_2^z$ ($\tau$ species). Further, $R$ should be invariant with respect to the interchange of $\sigma$'s and $\tau$'s.

The most general form of $R$ subject to the above may be written as

$$\begin{aligned} R_{12} = {} & g_0 + g_1\sigma_1^z\tau_1^z + g_2(\sigma_1^z\sigma_2^z + \tau_1^z\tau_2^z) + g_3\sigma_2^z\tau_2^z + g_4(\sigma_1^+\sigma_2^- + \tau_1^+\tau_2^- + \text{H.c.}) + g_5(\sigma_1^z\tau_2^z + \sigma_2^z\tau_1^z) \\ & + g_6\sigma_1^z\sigma_2^z\tau_1^z\tau_2^z + g_7[(\sigma_1^+\sigma_2^- - \sigma_2^+\sigma_1^-)\tau_1^z + (\tau_1^+\tau_2^- - \tau_2^+\tau_1^-)\sigma_1^z] + g_8(\sigma_1^+\sigma_2^- + \text{H.c.})(\tau_1^+\tau_2^- + \text{H.c.}) \\ & + g_9[(\sigma_1^+\sigma_2^- - \sigma_2^+\sigma_1^-)\tau_2^z + (\tau_1^+\tau_2^- - \tau_2^+\tau_1^-)\sigma_2^z] + g_{10}[(\sigma_1^+\sigma_2^- + \text{H.c.})\tau_1^z\tau_2^z + (\tau_1^+\tau_2^- + \text{H.c.})\sigma_1^z\sigma_2^z] \\ & + g_{11}(\sigma_1^+\sigma_2^- - \sigma_2^+\sigma_1^-)(\tau_1^+\tau_2^- - \tau_2^+\tau_1^-). \end{aligned} \quad (7)$$

We have twelve parameters $g_0, \ldots, g_{11}$ at our disposal and wish to satisfy the (linear) operator equation (3).

The algebra involved is tedious. I used the symbolic manipulation package REDUCE2 in order to perform the calculations on a computer. The problem and the method used to solve it seem to be general enough to justify some discussion. The basic idea used was to convert Eq. (3) into a partial differential equation, by the use of a representation of spin operators as partial derivatives on the space of polynomials.[5,6] [Such a representation is natural[6] if we construct unnormalized spin coherent-states $|z\rangle = \exp(zs^-)|\uparrow\rangle$ and represent abstract spinors by "wave functions" $\psi(z) = \langle z|\psi\rangle$.] Specifically for $s = \frac{1}{2}$, at a site $i$, we may write $\sigma_i^+ \to \partial/\partial x_i$, $\sigma_i^- \to x_i - x_i^2\,\partial/\partial x_i$, $\sigma_i^Z \to 1 - 2x_i\,\partial/\partial x_i$, and the wave functions $|\uparrow\rangle \to 1$, $|\downarrow\rangle \to x_i$. Associating $x_1, x_2, x_3$ with the three $\sigma$'s and $y_1, y_2, y_3$ with the three $\tau$'s, we see that manifold of 64 states is spanned by the wave functions $\pi(x_i)^{n_i}\pi(y_i)^{m_i}$, with $n_i, m_i = 0, 1$. We use the symmetries mentioned above, and the transformations generated by $\sigma_1^x\sigma_2^x$ or $\tau_1^x\tau_2^x$, which reverse all arrows on one sublattice, thereby negating the "fields" $h_1$ and $h_2$, and negate all $g_n$'s with $n$ odd. This enables us to restrict the considerations to the nine (irreducible) wave functions $x_1, x_2, x_3$, $x_1y_1$, $x_1y_2$, $x_1y_3$, $x_2y_2$, $x_2y_3$, and $x_3y_3$. The Yang-Baxter operator [left-hand side of Eq. (3) minus the right-hand side] is applied to these states and the coefficients of all the states in the resultant are required to vanish. Also the conjugates ($h_i \to -h_i$, $g_{2n+1} \to -g_{2n+1}$) are required to vanish. A total of 144 linear homogeneous equations result. Eleven of them suffice to determine the $g$'s: This is an overdetermined set of linear equations. By explicit calculation, I checked that all the expressions vanish identically with a choice of parameters.

Let us define $a_3 \equiv a_1a_2 + b_1b_2$, $a_4 \equiv a_1a_2 - b_1b_2$, $b_3 \equiv b_2a_1 - b_1a_2$, $b_4 \equiv b_2a_1 + b_1a_2$, $s_+ \equiv \sinh(h_2+h_1)$, $s_- \equiv \sinh(h_2 - h_1)$, $c_+ \equiv \cosh(h_2+h_1)$, $c_- \equiv \cosh(h_2 - h_1)$. Further, let $K_1 \equiv g_0 - g_6$ and $K_2 \equiv g_4 - g_{10}$. In terms of these, we find the relations $g_3 = g_1$, $g_9 = g_7$,

$$\begin{aligned} & g_7 = \tfrac{1}{2}s_- b_4K_1; \quad g_8 = c_+b_3K_2; \quad g_{11} = s_+b_4K_2; \\ & g_4 + g_{10} = c_-b_3K_1; \quad g_1 + g_5 = \tfrac{1}{2}s_-a_4K_1; \quad g_1 - g_5 = \tfrac{1}{2}s_+b_3K_2/(b_2^2 - b_1^2); \\ & g_0 + g_6 + 2g_2 = c_-a_3K_1; \quad g_0 + g_6 - 2g_2 = c_+b_4K_2/(b_2^2 - b_1^2). \end{aligned} \quad (8)$$

These ten equations enable us to express all the $g$'s in terms of $K_1$ and $K_2$. The final equation reads

$$\frac{K_2}{K_1}=\frac{\sinh(h_2-h_1)}{\sinh(h_2+h_1)}\frac{(b_2^2-b_1^2)}{(a_2b_2-a_1b_1)}=\frac{\cosh(h_2-h_1)}{\cosh(h_2+h_1)}\frac{(b_2^2-b_1^2)}{(a_2b_2+a_1b_1)}. \tag{9}$$

Consistency requires a single constraint (obtained by equating the two expressions),

$$a_1b_1/a_2b_2=\sinh(2h_1)/\sinh(2h_2). \tag{10}$$

This condition, together with the free Fermi condition ($a_1^2+b_1^2=1=a_2^2+b_2^2$) is then sufficient to guarantee the commutation of the two transfer matrices. From Eq. (2) we see that Eq. (10) is not a new constraint; it is automatically fulfilled if we require that the two transfer matrices commute with $H$ [i.e., Eq. (2)]. Thus, the one-parameter family found by requiring commutation with $H$ is fundamental. This situation is identical to the one encountered in the eight-vertex model.[3,7]

The structure of the $R$ operator is rather unwieldy and is best summarized in the following equations, describing its operation on the set of states $\psi_0=1$, $\psi_1=x_1$, $\psi_2=x_2$, $\psi_3=x_1y_1$, $\psi_4=x_1y_2$, $\psi_5=x_2y_2$, $\psi_6=x_2y_1$:

$$R\psi_0=K_1(c_-a_3+s_-a_4)\psi_0,\quad R\psi_1=K_1\psi_1+K_1(c_-b_3+s_-b_4)\psi_2,\quad R\psi_2=K_1\psi_2+K_1(c_-b_3-s_-b_4)\psi_1,$$

$$R\psi_3=\frac{K_2}{b_2^2-b_1^2}(c_+b_4+s_+b_3)\psi_3+K_2(\psi_4+\psi_6)+K_2(c_+b_3+s_+b_4)\psi_5,$$

$$R\psi_4=K_2(\psi_3+\psi_5)+\frac{K_2}{b_2^2-b_1^2}(c_+b_4-s_+b_3)\psi_4+K_2(c_+b_3-s_+b_4)\psi_6, \tag{11}$$

$$R\psi_5=K_2(c_+b_3+s_+b_4)\psi_3+K_2(\psi_4+\psi_6)+\frac{K_2}{b_2^2-b_1^2}(c_+b_4+s_+b_3)\psi_5.$$

The other equations may be inferred by use of appropriate symmetries. Let us note the symmetry of $R$; $R_{21}=R_{12}^{\dagger}$.

Finally, I remark on a curious side result. If we rotate the lattice through $\pi/2$ (and exchanging $a\leftrightarrow b$), the column-to-column transfer matrix may be shown to commute with another Hamiltonian. Linearizing Eq. (3) about $\theta_2=\theta_1$ we note that the first-order term in the variation has the form of Sutherland[7] for the commutation of the transfer matrix (column-to-column) with a Hamiltonian. The latter is the sum of two-body terms, each of the form of $R_{12}$ [Eq. (7)] with the first-order variations of the $g$'s substituted. The Hamiltonian is not Hermitean ($\delta g_7\neq 0$), but appears to be intimately related to the Hubbard model.

The work reported here should be useful in understanding further the properties of the 1D Hubbard model. Generalization of the statistical model to allow for the eight-vertex configurations obeying the free-terms condition seems interesting.

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