One-Dimensional Fractional Quantized Hall Effect

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The fractional quantized Hall effect in channel geometries with a narrow width is discussed. Analytic arguments as well as finite-cluster calculations are presented that suggest that a gap exists in the excitation spectrum for even- as well as for odd-denominator filling factors. Possible ways that one might sort this from localization effects in experimental measurements are discussed.

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There has been much interest in the fractional quantized Hall effect (FQHE) recently. Most of the experiments are carried out on interfaces between semiconductors.[1] Lately, it has become possible to construct channel structures as narrow as 100 Å on metal-oxide-semiconductor field-effect transitors.[2] It is thus natural to ask if it is possible to observe the FQHE in these structures. Investigation of these effects will not only provide for some new interesting phenomena but also further our understanding of the FQHE in general. From analytic arguments as well as small-cluster calculations, I find that, if the channel width is narrow enough, a gap exists in the excitation spectrum for even- as well as for odd-denominator filling factors. In contrast, in the 2D case, a gap exists in the excitation spectrum only for odd-denominator filling factors.

There are different ways by which this result can be understood. Perhaps the clearest of these starts off from a "one dimensional" description of the FQHE.[3]

In both the Landau and the circular gauge, the single-particle basis function is specified by a one-dimensional label, the y momentum j on the one hand and the angular momentum m on the other. The total angular momentum M is a constant of motion in one case while the total y momentum J is also a constant of motion in the other. In general, the discussion in one gauge can be translated easily into the other gauge. In this paper, we shall use the language of the Landau gauge. The basis set can be written as product wave functions of Landau orbitals given by[4]

$$\phi_j(\mathbf{r}) = \exp[ix_j y - \tfrac{1}{2}(x-x_j)^2]/(\pi^{1/2}L_y)^{1/2}, \quad (1)$$

$$x_j = (2\pi/L_y)j. \quad (2)$$

L_y is the width in the y direction. If we take the y axis to be oriented across the channel, then the separation between the basis functions, Δx_j, increases as the channel width L_y is decreased. The Hamiltonian in second quantized form can be written, except for trivial constants, as

$$H = \sum_{[j]} A(j_1,j_2,j_3,j_4) C_{j_1}^\dagger C_{j_2}^\dagger C_{j_3} C_{j_4}. \quad (3)$$

The A's are integrals of the Coulomb potential and the Landau orbitals ϕ_j:

$$A(j_a,j_b',j_b,j_a') = \int d^2r\, d^2r'\, \phi_{j_a}^*(\mathbf{r})\phi_{j_b}^*(\mathbf{r}') \times \phi_{j_b}(\mathbf{r})\phi_{j_a'}(\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|. \quad (4)$$

There is a similarity between Eq. (3) and the Hubbard model. To bring out this similarity and to gain more insight, we decompose the Hamiltonian as a sum of a diagonal, H_d, and an off-diagonal, H_o , term:

$$H = H_d + H_o, \quad (5)$$

$$H_d = \sum_{k,j} V(k) n_j n_{j+k}, \quad (6)$$

$$H_o = \sum_{l=1}\sum_{k=1} t_l(k) \sum_i C_{i-l}^\dagger C_i C_{i+k} C_{i+k+l}^\dagger + \text{c.c.} \quad (7)$$

The t's are the hopping integrals. To illustrate the behavior of these parameters, I show $V(k)$ and $t_1(k)$ as a function of the distance k for the twelve-site case for different values of the aspect ratio $L_x/L_y = \gamma$ in Figs. 1 and 2.

The 2D case corresponds to γ close to 1. I first recapitulate the nature of the commensuration energy in the 2D case. When $\gamma = 1$, $V(k)$ attains a maximum and then decreases as one approaches the origin. This comes from the exchange. More precisely,

$$V(k) = 2[A(j_{23}=0, j_{13}=k) - A(j_{23}=k, j_{13}=0)], \quad (8)$$

where $j_{ab} = j_a - j_b$; the first term (second) is the direct (exchange) contribution. At small distances k, these two terms are comparable in magnitude, and the net value of V is reduced. As k increases, the exchange contribution dies off exponentially fast and only the first term remains. The distance l_t at which V turns from repulsive to attractive is independent of the sample size N_s. The diagonal term H_d exhibits two local minima. Because of the attractive part of V, the particles have a tendency to form clusters of size l_0 less than l_t with total y momentum J_c. For example, in the case of $\frac{1}{3}$-filled Landau level, the particles tend to form clusters of two in the twelve-site case and clus-

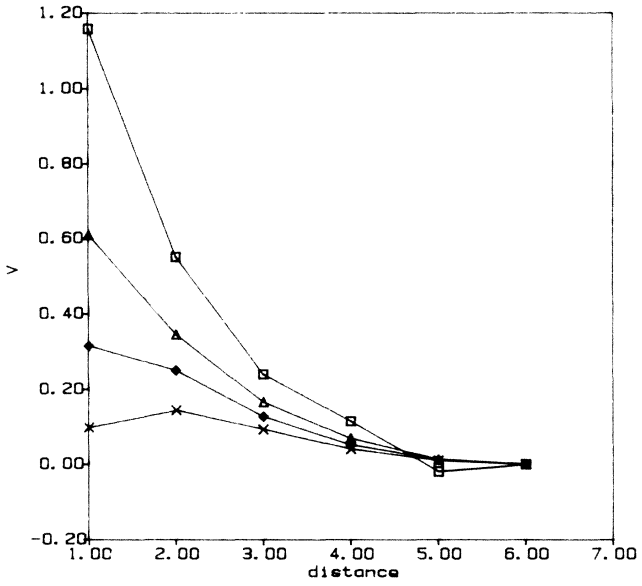

FIG. 1. The effective repulsion between the electrons at different distances for aspect ratios $r=12$ (squares), $r=4$ (triangles), $r=2$ (diamonds), and $r=1$ (crosses) for a twelve-site system.

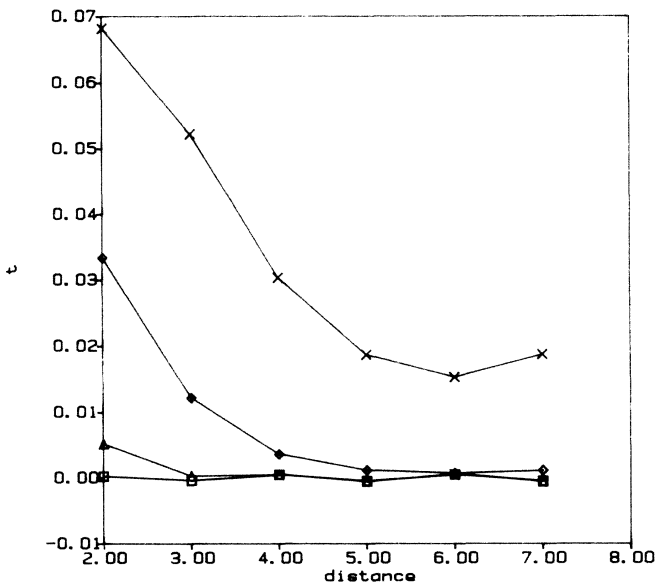

FIG. 2. The effective hopping matrix elements of the electrons at different distances for aspect ratios $r=12$ (squares), $r=4$ (triangles), $r=2$ (diamonds), and $r=1$ (crosses) for a twelve-site system.

ters of four in the 48-site case. On the other hand, a uniform distribution with total y momentum J_u is favored by the direct energy acting together with the hopping term. Because of the hopping term, the energy will be lowered if the wave function forms a linear combination from these two minima. However, a linear combination can be formed just for odd denominators because only then are the y momenta of the two minima the same. For even denominators, because of the competition from the two local minima, we obtain degenerate ground states with the momenta close to those for the cluster and uniform distributions.

The 1D limit is approached when γ becomes large. As γ is increased the separation between the basis functions is increased, and the overlap of the wave functions is decreased. As a consequence, the hopping matrix element t, as well as the exchange contribution to V, is decreased. This is obvious from Fig. 1 where we see that for a γ of 12, V no longer decreases as the origin is approached. Furthermore, from Fig. 2, the hopping matrix element t_1 has decreased to a magnitude of 10^{-3}. Aside from the fact that the hopping term is a two-particle operator, this is not that different from the ordinary Hubbard model. To further confirm this, I have diagonalized the Hamiltonian numerically for the twelve-site case. When $\gamma=12$ I find gaps equal to 0.691 and 0.38 in the $\frac{1}{2}$-filled and the $\frac{1}{3}$-filled cases, respectively; these correspond to $V(1)-V(2)=0.7$ and $V(2)-V(3)=0.32$, as we physically expected. In contrast to the 2D situation where the gap for $\frac{1}{3}$ filling is much bigger than that for $\frac{1}{2}$ filling (no gap), the situation is reversed here. Recall that the gap is approximately equal to 0.1 for the $\frac{1}{3}$-filled case in the 2D situation. The magnitude of the gap seems bigger in the present case. When $\gamma=4$, the gap is equal to 0.31 at $\frac{1}{2}$ filling, whereas $V(1)-V(2)=0.27$. For $\gamma=2$, I find no clear evidence of a gap.

To gain more understanding of the nature of the ground state, I have also looked at the projected static structure factor $\bar{S}(k)$, obtained from the ordinary structure by demanding that all intermediate states lie in the lowest Landau level. This is defined to be

$$\bar{S}(k)=\langle\bar{\rho}_{-k}\bar{\rho}_k\rangle, \tag{9}$$

where $\bar{\rho}_k$ is the projected density operator defined by

$$\bar{\rho}_k=\sum_j \exp(ik\,\partial_{z_j})\exp(\tfrac{1}{2}ik^*z_j). \tag{10}$$

where $z_{(i,j)}=x_{(i,j)}+iy_{(i,j)}$. For large k, $S(k)$ approaches 1 whereas $\bar{S}(k)$ approaches 0. I find that for the case with $\gamma=4$ there is a giant peak of magnitude very close to 6 (the maximum possible value for a six-particle system) at $q=\pi/a$, where a is the intersite spacing, indicating the formation of the "uniform" distribution with every other site occupied in the half-filled case. For other values of q its magnitude is less than 10^{-3}. Hence one has nearly perfect order. The magnitude of this peak is about 5 times larger than that from finite-cluster calculations for the $\frac{1}{3}$-filled case and 10 times larger than that from the Laughlin wave function.

In general, the picture of the ground state is thus

one of an average occupation of every ν^{-1} site for a filling factor $1/\nu$. If $t(1/\nu)$ is small compared with $V(1/\nu)-V(1/\nu-1)$, a perturbation expansion in the ratio of these two quantities will provide a reasonable estimate of the gap. A more interesting question concerns the existence of a gap in the limit that the ratio is not small. For the ordinary 1D Hubbard model, a gap exists no matter what the value of $t/\Delta V$ is. If one can approximate the two-particle hopping term H_o by

$$H_o=\sum_{l=1}\left[\sum_{k=1} t_l(k)\langle C_{i+k}C^{\dagger}_{i+k+l}\rangle\right]\sum_i C^{\dagger}_{i-l}C_i + \text{c.c.}, \quad (11)$$

then the usual arguments for the Hubbard model can be applied. The validity of this approximation has not been investigated in detail, however.

To investigate a possible size dependence of the Hamiltonian parameters, I calculated $V(1)-V(2)$ for different sizes N_s but with the same width L_y. I found that for $L_y=(6\pi)^{1/2}$ it is equal to 0.27, 0.32, and 0.32 for $N_s=12$, 24, and 48, respectively. The hopping matrix elements $t_1(1)$ is of the order of 10^{-2} for all these cases. For $N_s=48$, I found that the diagonal term $V(1)-V(2)$ (0.21) is still substantially bigger than the hopping matrix element (0.02) up to $L_y=(8\pi)^{1/2}$.

The numerical calculations were carried out for periodic boundary conditions in the y direction. This differs from the actual boundary condition. The spread of the wave function in the lateral direction is $L_y=(2\pi N_s/\gamma)^{1/2}$. For $N_s=12$, $L_y\approx 5$ and 2 for $\gamma=4$ and 12, respectively. When L_y is small, boundary effects will be important and our Hamiltonian may not be accurate at $\gamma=12$. It is possible to employ a slightly different basis set of the form $\Sigma_k f(k-x_j)\psi_j$ so that the wave function dies off in the lateral direction away from the channel. An example of f would be $1/(k^2+\lambda^2)$ for some parameter λ. If the width L_y is of the order of the Larmor radius, by properly chosing f one can construct wave functions located at $\mathbf{R}$ of the form $\psi=\exp[-|\mathbf{r}-\mathbf{R}_j|^2/4+i\hat{\mathbf{z}}\times R\cdot\mathbf{r}/2]$ which appears in the circular gauge. We thus expect a slight smearing of the wave function in the x direction. To estimate the possible changes to the Hamiltonian, I assumed a periodic arrangement of the wave function ψ and found that the direct contribution to $V(1)-V(2)$ is given approximately (I have ignored the image contributions) by

$$\tfrac{1}{2}\pi^{1/2}[\exp(-\tfrac{1}{8}R_1^2)I_0(\tfrac{1}{8}R_1^2) - \exp(-\tfrac{1}{8}R_2^2)I_0(\tfrac{1}{8}R_2^2)],$$

whereas the exchange contribution is given by

$$\tfrac{1}{2}\pi^{1/2}[\exp(-\tfrac{3}{8}R_1^2)I_0(\tfrac{1}{8}R_1^2) - \exp(-\tfrac{3}{8}R_2^2)I_0(\tfrac{1}{8}R_2^2)].$$

We hence obtain $V(1)-V(2)=0.16$ for $\gamma=12$. The hopping matrix element is given by

$$t(1)=\tfrac{1}{2}\pi^{1/2}[\exp(-\tfrac{1}{2}R_1^2)-\exp(-\tfrac{1}{2}R_2^2)].$$

$t=0.04$ for $\gamma=12$. Thus $V(1)-V(2)$ is still positive and much bigger than $t(1)$ in this case. Because of the change in boundary condition, the single-particle energy will also be shifted away from the value $0.5\hbar\omega_c$. If the channel width is not uniform, this can produce a random shift in this energy and create additional randomness in the system. If the fluctuation of this shift is less than the gap I do not expect a drastic change in the results obtained here.

There are other explanations of the commensuration condition for the 2D situation. It is not difficult to understand why they are not applicable in the present case. Laughlin[5] proposed a trail wave function of the form

$$\psi_{\mathrm{L}}=\prod_{i>j}(z_i-z_j)^m\exp\left(-\tfrac{1}{4}\sum_i r_i^2\right), \quad (12)$$

with probability density

$$|\psi_{\mathrm{L}}|^2=\exp\left(2m\sum_{i>j}\ln r_{ij}-\tfrac{1}{2}\sum_i r_i^2\right); \quad (13)$$

r_i is the distance of the point (x_i,y_i) from some origin of coordinates, and r_{ij} is the distance between the points (x_i,y_i) and (x_j,y_j). This density is identical to the partition function of a one-component plasma with m corresponding to the effective coupling of the plasma. For this plasma to stay together, m is determined by the density and is equal to the inverse filling factor. One might attempt to generalize this formula to the present case. Imagine the electrons being confined to a large ring with a fixed radius C. Then one can in principle easily modify the above formula by multiplying it by $\Pi_j z_j^a$ for some constant a so that the mean magnitude of z is C. The probability density then corresponds to the partition function of a 1D plasma. The thermodynamic properties of this 1D logarithmic plasma have been studied quite intensively in the context of random matrices. The free energy was first conjectured by Dyson and proved by Wilson.[6] They found that the free energy F is given by

$$F(2m)=L(m)/2m+\tfrac{1}{2}(1-\ln m), \quad (14)$$

where $L(z)=\ln\Gamma(1+z)$. I note that, at a given temperature m, F does not exhibit any minimum as a function of the density of the particles. Hence there is no corresponding relationship between the density and m here. Whereas in the 2D case there is the term Σr_i^2 that holds the particles together and one can determine m in terms of the density, in the present case no such term is present; one can basically choose any m.

Tosatti and Parrinelo[7] pointed out that multiparticle exchange on an uncorrelated Wigner lattice may lead

to a commensuration energy. For exchanges involving particles in a loop on a lattice of area Aa_0^2 (a_0 is the lattice constant; for the $\frac{1}{3}$-filled case, $a_0=4.665$) involving $N_{\|}$ particles, the exchange integral has a phase factor equal to $\frac{1}{2}Aa_0^2$. $\frac{1}{2}Aa_0^2$ is modulo $N_{\|}\pi/\nu$, where $1/\nu$ is the filling factor. Hence when ν is odd, these exchanges enhance the total energy. Chui, Hakim, and Ma[8] found that for correlated Wigner lattices, the commensuration condition becomes exact only when the area covered by the particles exchanged becomes large. Similar conditions seem to have been found in the path-integral calculation by Kivelson *et al.*[9] When the width of the channel is reduced, the possibilities of the exchanges are limited. One can imagine a ring geometry and focus on the exchange around the ring. A phase factor of $Aa_0^2/2$ will again be obtained. $Aa_0^2/2$ is, however, no longer modulo $N_{\|}\pi/\nu$ and the normal argument is inapplicable. These arguments did bring out the question of possible new physics of ring geometries associated with the Bohm-Aharonov effect which will be discussed separately.

What is the critical width of the channel so that 2D-type behavior is recovered? The present numerical calculation indicates that 1D behavior is observed up to a width of the order of 5 Larmor radii. However, we do not know what the maximum width can be.

We next look at the question of impurity pinning. Because of the presence of the gap, when the impurity potential is weak, these impurities do not couple the ground state strongly to the excited states. Analogous to the 2D case, the ground states possess a center-of-mass degeneracy and the impurities can mix them in principle. Let us call two of these states with different mean values of center of mass $|a\rangle$ and $|b\rangle$. In both the 2D and the 1D case, they are linear combinations of products of single-particle wave functions. The separation between the single-particle mean spacing of these degenerate states is of the order $1/L_y$. We expect the matrix element $\langle a|V_i(r_j)|b\rangle$ to be equal to 0 for any single-particle impurity potential V_i because the overlap of the wave function of the remaining particles $l\neq j$ goes to zero in the thermodynamic limit. If the impurities are randomly arranged, we expect $\langle a|\sum_j V_i(j)|a\rangle = \langle b|\sum_j V_i(j)|b\rangle$. Hence we expect the ground state not to be pinned if the impurities are weak.

How can these effects be observed experimentally? The most direct observation would be to measure ρ_{xy} as a function of the filling factor. It may be difficult to put leads onto structures that are 100 Å wide. One possibility is to do an ac-type measurement so that no external leads are necessary. Another possibility is to try to measure ρ_{xx}. However, it may be masked by possible localization effects of the electrons that have not been discussed so far. One can follow Halperin[10] to argue that the system may not be localized for a weak enough impurity potential; it is not clear if this weak-scattering condition can be satisfied experimentally, however. Since localization effects by themselves are not a strong function of the filling factor, one may try to look for changes as the filling factor is changed. Alternatively, one may manufacture a "weak link" of a narrow channel of width w and length l connecting two 2D regions. At even-denominator filling factors, the 2D regions will not be affected. If we pick the length l to be smaller than the localization length, then this weak link may act like a switch and turn the current off at these situations.

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