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Curious Properties of Quantum Ensembles Which Have Been Both Preselected and Post-Selected

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A recent result concerning the assignment of well-defined values to certain noncommuting observables in quantum ensembles which have been both preselected and post-selected appears to show that quantum mechanics is "contextual." Extending the argument to the case of measurements on two separated spin-$\frac{1}{2}$ systems suggests that quantum mechanics is nonlocal. The significance of these phenomena is evaluated, and it is shown that they have no bearing on the question of contextuality or nonlocality.

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In a recent publication,[1] Albert, Aharonov, and D'Amato exploit an expression for the probability of events between measurements first derived by Aharonov, Bergmann, and Lebowitz[2] to provide answers to certain questions concerning preselected and post-selected ensembles within the framework of orthodox quantum mechanics. Suppose that an initial "nondemolition" measurement of a complete commuting set of observables M_i on a system yields the state ψ_I, and a final measurement (a certain time interval later) of a different complete commuting set M_F on the same system yields the state ψ_F, where ψ_I and ψ_F lie in two orthogonal eigenplanes which intersect in an eigenvector α_k corresponding to the eigenvalue m_k of a complete commuting set M. Provided that neither ψ_I nor ψ_F is orthogonal to α_k, a measurement of M performed in the time interval between the measurements of M_I and M_F would yield the result m_k with certainty since, according to the Aharonov-Bergmann-Lebowitz probability rule,

$$\text{prob}_M(m_j) = \frac{|(\alpha_j,\psi_I)|^2|(\psi_F,\alpha_j)|^2}{\sum_i|(\alpha_i,\psi_I)|^2|(\psi_F,\alpha_i)|^2} \cdots . \quad (1)$$

This prediction of quantum mechanics can, of course, be confirmed by performing sequences of measurements $M_I \to M \to M_F$: All sequences in which the initial and final states are ψ_I and ψ_F will have the value m_k for M.[3]

For example, in a three-dimensional Hilbert space, suppose that the three orthogonal eigenvectors of M are α_1, α_2, α_3. These vectors define three orthogonal planes: P_{12}, P_{13}, P_{23}. If $\psi_I \in P_{12}$ $[(\psi_I,d_2)\neq 0]$ and $\psi_F \in P_{23}$ $[(\psi_F,\alpha_2)\neq 0]$ then a measurement of M performed between the measurements of M_I and M_F would yield the result m_2, corresponding to the eigenvector α_2, with certainty. This example is discussed by Albert, Aharonov, and D'Amato, who go on to show that a measurement of N instead of M, where N is associated with the eigenvectors β_1, $\beta_2=\alpha_2$, β_3 (i.e., the β basis is obtained by rotating the α basis about α_2 in the P_{13} plane), would *not* yield the result n_2 corresponding to α_2 with certainty. This is because $\text{prob}_N(n_1)\neq 0$, since $(\psi_I,\beta_1)\neq 0$, and $(\psi_F,\beta_1)\neq 0$, and $\text{prob}_N(n_3)\neq 0$, since $(\psi_I,\beta_3)\neq 0$ and $(\psi_F,\beta_3)\neq 0$, and clearly $\text{prob}_N(n_2)\neq 1$.

The discrepancy between M and N is the "curious new statistical prediction of quantum mechanics" referred to in the title of the Albert-Aharonov-D'Amato paper. The authors argue that this result calls into question a crucial assumption underlying the Kochen and Specker proof[4] that a hidden-variable theory of the quantum statistics is impossible, or the argument via Gleason's theorem discussed by Bell.[5] In the class of hidden-variable theories relevant to these "no-go" theorems, observables corresponding to self-adjoint Hilbert space operators are represented by real-valued random variables on a measure space parametrized by the values of the hidden variables. These proofs require that the hidden variables assign the same value to the random variable representing the projection operator P_{α_2}, whether this observable is measured via M or via N. In effect, the value assignment is required to satisfy a "meshing condition," that M has the value m_2 if and only if N has the value n_2. Bell pointed out that this assumption need not be satisfied at all points in the measure space of hidden variables, in order for a hidden-variable theory to generate the quantum statistics for measures corresponding to quantum states. Thus, these theorems do not exclude "contextual" hidden-variable theories, in which the hidden variables assign to P_{α_2} values (1 or 0) in the context of an M measurement which may differ from the values assigned to this observable in the context of an N measurement. In such theories, P_{α_2} is represented by a family of random variables, one for each complete commuting set of observables to which P_{α_2} belongs.

What Albert, Aharonov, and D'Amato appear to have shown is that *quantum mechanics itself is a contextual theory in this sense,* for the values of observables in ensembles which have been both preselected and post-selected on the basis of measurement results. That their argument is fallacious can be seen by noting that the subensemble of the preselected ψ_I ensemble that is post-selected for ψ_F via an intervening M measurement differs from the subensemble that is post-selected for ψ_F via an intervening N measurement. With ψ_I, ψ_F, M, and N as chosen by Albert, Aharonov, and D'Amato, one-quarter of the preselected ensemble is post-selected via the intervening M measurement (all *these* systems having yielded the value of m_2 for M), while three-eighths of the preselected ensemble is post-selected via the intervening N measurement (only *two-thirds* of *these* systems having yielded the value n_2 for N).[6] Thus, the notion of a statistical ensemble which is specified by preselection and post-selection via an arbitrary intervening measurement is not well defined in quantum mechanics. Put simply, systems initially in the state ψ_I which are subjected to an N measurement, and subsequently yield the state ψ_F after an M_F measurement, would not necessarily yield this final state if subjected to a measurement of M instead of N. Post-selecting for the eigenvector ψ_F of M_F after an M measurement is not the same thing as post-selecting for ψ_F after an N measurement.

It is possible to construct an example in a four-dimensional Hilbert space, analogous to the three-dimensional example of Albert, Aharonov, and D'Amato, in which the fraction of the preselected ψ_I ensemble that is post-selected for ψ_F is the same for different intervening measurements. In this case, with an appropriate choice of ψ_I, ψ_F, M, and N, quantum mechanics appears to be nonlocal.

To see this, consider the question which Bell posed,[7] whether the contextuality of hidden-variable theories necessarily extends to the case of spatially separated composite systems—say, two spin-$\frac{1}{2}$ systems, S and S', represented in a four-dimensional Hilbert space—in such a way as to violate locality. Specifically, suppose that we consider a measurement of the complete commuting set M on the composite system associated with the basis defined by the four vectors $\theta_1 = \alpha_1 \otimes \alpha_1'$, $\theta_2 = \alpha_1 \otimes \alpha_2'$, $\theta_3 = \alpha_2 \otimes \alpha_1'$, $\theta_4 = \alpha_2 \otimes \alpha_2'$, where α_1, α_2 are the eigenvectors of the spin component in some direction, say $\mathbf{a}$, for S, with corresponding eigenvalues a_1, a_2 (for spin "up" in the $\mathbf{a}$ direction, spin "down" in the $\mathbf{a}$ direction), and α_1', α_2' are the spin eigenvectors for S'. Thus, M is the set $\{J_a, J_a'\}$. Bell showed that any hidden-variable theory capable of reproducing the quantum statistical correlations for the separated systems would have to be nonlocal in the following sense: Suppose that we change the α' basis to β_1', β_2', corresponding to a measurement of the spin component in a new direction *on* S', generating a new basis, $\alpha_1 \otimes \beta_1'$, $\alpha_1 \otimes \beta_2'$, $\alpha_2 \otimes \beta_1'$, $\alpha_2 \otimes \beta_2'$, associated with a measurement of the complete commuting set $N = \{J_a, J_b'\}$ on the composite system. Then there must exist a discrepancy between M and N *with respect to the value of the spin component of* S for some values of the hidden variables. This result has been generalized in various ways to cover a wide class of hidden-variable theories, including stochastic hidden-variable theories.

Now, suppose that there existed a quantum state W characterizing the composite system $S + S'$, with the following property: If we were to measure spin in the $\mathbf{a}$ direction on S and spin in the $\mathbf{a}$ direction on S', then

$$\text{prob}_W(a_1) = 1,$$

but if we were to measure spin in the $\mathbf{a}$ direction on S and spin in some other direction $\mathbf{b}$ on S', then

$$\text{prob}_W(a_1) \neq 1.$$

If such a state existed, quantum mechanics would be a nonlocal theory, and pairs of systems prepared in this state could be used to communicate information superluminally. Of course, no such state exists. But we find precisely this situation for quantum ensembles which have been appropriately preselected and post-selected (and which cannot therefore be characterized by a unique quantum state). In this case superluminal communication is blocked because the probability assignments refer to the interval between measurements.

To construct the example, let M_I be the complete commuting set corresponding to the basis formed by the three vectors of the triplet state and the single vector of the singlet state: θ_1, $\frac{1}{2}\sqrt{2}(\theta_2+\theta_3)$, $\frac{1}{2}\sqrt{2}(\theta_2-\theta_3)$, θ_4, i.e., M_I consists of the operators $\{J^2[S+S'],\ J_a[S+S']\}$ for the composite system. Let ψ_I be the singlet state, i.e.,

$$\psi_I=\tfrac{1}{2}\sqrt{2}(\theta_2-\theta_3)\in P_{23},$$

where P_{23} denotes the plane spanned by θ_2, θ_3.

Let M_F be the complete commuting set corresponding to the basis $\gamma_1\otimes\alpha_1'\in P_{13}$, $\alpha_1\otimes\alpha_2'\in P_{24}$, $\alpha_2\otimes\alpha_1'\in P_{13}$, $\gamma_2\otimes\alpha_2'\in P_{24}$, where the γ basis corresponds to a measurement of the spin component in a new direction *on S*, i.e., $M_F=\{J_c, J_{a'}\}$. Let

$$\psi_F=\gamma_1\otimes\alpha_2'\in P_{24}.$$

Now suppose that S and S' separate in space and consider M and N as above, where N differs from M only with respect to the direction of the spin measurement *on S'*. Note that $\alpha_1\otimes\beta_1'\in P_{12}$, $\alpha_1\otimes\beta_2'\in P_{12}$, $\alpha_2\otimes\beta_1'\in P_{34}$, $\alpha_2\otimes\beta_2'\in P_{34}$.

Since $\psi_I\in P_{23}$ and $\psi_F\in P_{24}$, it follows that $\mathrm{prob}_M(m_2)=1$, since $\mathrm{prob}_M(m_k)=0$ for $k=1,3,4$, by (1). But this means that

$$\mathrm{prob}_M(a_1)=1,$$

i.e., the system S would register a spin component up in the **a** direction with certainty for a measurement of M performed in the interval between the measurements of M_I yielding ψ_I and M_F yielding ψ_F.

Now consider the measurement of N. By (1) we no longer have $\mathrm{prob}_N(n_k)=0$ for $k=3,4$, and so $\mathrm{prob}_N(a_2)\neq 0$ and

$$\mathrm{prob}_N(a_1)\neq 1,$$

i.e., the system S would not register a component up in the **a** direction with certainty for a measurement of N performed in the interval between the measurements of M_I yielding ψ_I and M_F yielding ψ_F.

If we choose

$$\beta_1'=\tfrac{1}{2}\sqrt{2}\alpha_1'+\tfrac{1}{2}\sqrt{2}\alpha_2',$$

$$\beta_2'=\tfrac{1}{2}\sqrt{2}\alpha_1'-\tfrac{1}{2}\sqrt{2}\alpha_2',$$

$$\gamma_1=\tfrac{1}{2}\sqrt{2}\alpha_1+\tfrac{1}{2}\sqrt{2}\alpha_2,$$

$$\gamma_2=\tfrac{1}{2}\sqrt{2}\alpha_1-\tfrac{1}{2}\sqrt{2}\alpha_2,$$

then one-quarter of the preselected ensemble is post-selected for ψ_F via either intervening measurement, so that we even have a case in which the ensembles characterized by the same preselection and post-selection but different intervening measurements (of M or N) are not manifestly different, yet $\mathrm{prob}_M(a_1)=1$ and $\mathrm{prob}_N(a_1)=\frac{1}{2}$.

Apparently, there exist ensembles which have been both preselected and post-selected on the basis of appropriate measurement results, with the curious property that whether or not a system registers spin up in a certain direction on measurement depends on the direction in which spin is measured on a system S' to which it is coupled, even if S and S' are separated by a spacelike interval. Does this mean that quantum mechanics is a nonlocal theory?

Now, although the fraction of the ensemble preselected for ψ_I which is eventually post-selected for ψ_F after an intervening M measurement is the same—one quarter—as the fraction which is post-selected for ψ_F after an intervening N measurement, there is no warrant for the inference that these fractions represent the same subensemble of the original preselected ensemble. If M is measured on the preselected ensemble specified by ψ_I, half the systems make a transition to the state $\alpha_1\otimes\alpha_2'$, and half of these make the transition to the post-selected state ψ_F. No other systems are post-selected for ψ_F. Call this ensemble E; it is a subensemble of the preselected ensemble characterized by a condition concerning subsequent transitions in a sequence of measurements of M and M_F; viz., $\psi_I\rightarrow\alpha_1\otimes\alpha_2'\rightarrow\psi_F$. If N is measured instead of M, one-quarter of the systems make a transition to the state $\alpha_i\otimes\beta_j'$ $(i=1,2;\ j=1,2)$ and one quarter of these, for each i and j, make the transition to the post-selected state ψ_F. Call this ensemble E'; it is a subensemble of the preselected ensemble characterized by a condition concerning subsequent transitions in a sequence of measurements of N and M_F, viz., the transitions $\psi_I\rightarrow\alpha_i\otimes\beta_j'$ $(i=1,2;\ j=1,2)$ $\rightarrow\psi_F$. The argument for nonlocality does not go through unless $E=E'$, and there is no reason to suppose that this is the case.

Conclusion—The somewhat curious analogs of contextuality and nonlocality which arise in the statistics of quantum ensembles which have been preselected and post-selected via an arbitrary intervening measurement have their origin in the fact that such ensembles are not well defined without specification of the intervening measurement. In particular, these phenomena have no bearing on the question of the contextuality of nonlocality of hidden-variable reconstructions of the quantum statistics.

[1]D. Z. Albert, Y. Aharonov, and S. D'Amato, Phys. Rev.

Lett. **54**, 5 (1985).

[2]Y. Aharonov, P. G. Bergmann, and J. L. Lebowitz, Phys. Rev. **134**, B1410 (1964).

[3]It is easy to see that these conditions on ψ_I and ψ_F are both necessary and sufficient for a "middle measurement" of M to yield m_k with certainty: The only way we can get $\text{prob}_M(m_k) = 1$ via the Aharonov-Bergmann-Lebowitz probability rule is if $\psi_I \in P_{ki}[(\psi_I, \alpha_K) \neq 0]$ and $\psi_F \in P_{kj}[(\psi_F, \alpha_K) \neq 0]$, where P_{ki} and P_{kj} are eigenplanes of M and intersect in α_K.

[4]S. Kochen and E. P. Specker, J. Math. Mech. **17**, 59 (1967).

[5]J. S. Bell, Rev. Mod. Phys. **38**, 447 (1966).

[6]It might be supposed that the discrepancy between M and N is inconsistent with the "meshing condition" in a hidden-variable theory of the quantum statistics. But this is not the case. For an intervening N measurement, there are three routes the systems can take: $\psi_I \rightarrow \beta_i \rightarrow \psi_F$, $i = 1, 2, 3$. The route $\psi_I \rightarrow \beta_2 = \alpha_2 \rightarrow \psi_F$ is taken by one-quarter of the preselected ensemble, the other two routes by one-sixteenth each. For an intervening M measurement, there is only one route the systems can take: $\psi_I \rightarrow \alpha_2 \rightarrow \psi_F$, and this route is taken by one-quarter of the preselected ensemble. The meshing condition implies that all the systems which take the route $\psi_I \rightarrow \beta_2 = \alpha_2 \rightarrow \psi_F$ for an intervening N measurement would necessarily take the route $\psi_I \rightarrow \alpha_2 \rightarrow \psi_F$, were M being measured instead of N, and this is quite consistent with the statistics.

[7]Bell posed the question in Rev. Mod. Phys. **38**, 447 (1966), and answered it in Physics 1, 195 (1965).