Location of Renormalization-Group Fixed Points

Swendsen[1] has presented a method of optimizing real-space renormalization-group (RG) transformations which provided improved convergence in estimating critical exponents for the $d=3$ Ising model. On this basis he suggested[1] that by proper choice of a RG kernel, "it is possible to place the fixed point *anywhere* on the critical hypersurface," arguing "that this does not conflict with the association of irrelevant operators with corrections to scaling." In this Comment we show that these assertions are, in general, incorrect. (It is *not* our intent, however, to criticize any of Swendsen's numerical results or to take issue with the effectiveness of the optimized transformations he actually implements.)

Consider, for simplicity, the RG formulation for a family of systems, like ferromagnetic Ising models in zero field, with only one relevant scaling variable. We show, first, that for a given renormalization group $\mathscr{R}$ there is a one-dimensional locus in the parameter space, the "principal trajectory" Π, along with there are no singular corrections[2] to a pure power law when the critical manifold is approached. The argument is most easily grasped when a two-dimensional field space, (t,u), suffices to describe the RG flows and $t=t_c(u)$ describes the critical manifold: One may regard t as the temperature and u as a secondary variable modifying the Hamiltonian. For a spatial rescaling factor $b\equiv e^l$, the flow equations for t, u, and a thermodynamic function $G(t,u)$, say a diverging susceptibility, are generically of the form[2] $dt/dl=Q(t,u)$, $du/dl=R(t,u)$, and $dG/dl=P(t,u)G+U(t,u)$, where U, P, Q, and R are differentiable functions. At a fixed point (t^*,u^*) of $\mathscr{R}$, which we suppose is the only one in the region of interest, one has $Q(t^*,u^*)=R(t^*,u^*)=0$. On linearizing[2,3] about (t^*,u^*), it is straightforward to prove[2,4] that, in general, G is asymptotically of the scaling form

$$G(t,u)\approx|\tilde{t}|^{-\gamma}X(\tilde{u}|\tilde{t}|^{\theta})+G_0, \tag{1}$$

where (i) $\tilde{t}$ and $\tilde{u}$ are *definite,* independent linear combinations of the deviations $\Delta t=t-t^*$ and $\Delta u=u-u^*$, (ii) γ is the exponent for G, and (iii) the correction-to-scaling exponent, θ, is positive since, by hypothesis, $\tilde{u}$ is an irrelevant variable. To ensure the analyticity of $G(t,u)$ away from criticality, the scaling function, $X(y)$, must be a smooth function[2,4] with an expansion $X_0+X_1y+\ldots$: In general, therefore, $G(t,u)$ contains a nonanalytic correction of the form $X_1|\tilde{t}|^{-\gamma}y=X_1\tilde{u}|\tilde{t}|^{-\gamma+\theta}\propto\tilde{u}|t-t_c(u)|^{-\gamma+\theta}$, where, near the fixed point, $t_c(u)$ is determined by $\tilde{t}=0$. Evidently, this leading singular correction vanishes *on but,* in general, *only on* the special locus $\tilde{u}=0$, which intersects the critical manifold *at* (and *only* at) the fixed point: This locus, Π, is just the unstable (or outgoing) trajectory emerging from (t^*,u^*).

Now consider a second RG, say $\hat{\mathscr{R}}$, operating in the same space and having a single fixed point $(\hat{t}^*,\hat{u}^*)$ which *differs* from[5] (t^*,u^*). The new principal trajectory, $\hat{\Pi}$, cannot be Π: Thus a parallel analysis for $\hat{\mathscr{R}}$ predicts the *non*vanishing of the $|\tilde{t}|^{-\gamma+\theta}$ corrections to $G\sim|\tilde{t}|^{-\gamma}$ along the original trajectory, Π. But the thermodynamic function $G(t,u)$ must have a unique functional depedence along any path in (t,u) space irrespective of what RG is used to analyze the system. One must thus conclude that any (nonsingular) renormalization group fitting the problem will *share* the *same* (nontrivial) fixed points and, indeed, the same scaling axes, $\tilde{t}=0$ and $\tilde{u}=0$, also.

The corrections induced by $\tilde{u}$ may be wholly analytic if θ is an *integer* (although, more typically, $\log|\tilde{t}|$ factors will enter[2]); it should *then* be possible to move the fixed point on the critical manifold. The same may (but usually will *not*) be true when $\theta=0$ and so a marginal variable is involved. The extension to further variables $u_2,u_3,\ldots$ does not alter the situation[2]: *On,* but *only on* the trajectory, Π, leaving the fixed point all singular corrections vanish. However, *redundant variables*[6] should be recalled. These represent reparametrizations of the basic fields which cannot change the physics (since the fields are integrated out in defining the partition function). One fixed point may be mapped into another by a change of redundant variables and two RGs, say $\mathscr{R}$ and $\hat{\mathscr{R}}$, may produce *formally* different fixed points by this mechanism. Nevertheless, a general point on a critical manifold *cannot* be transformed into a fixed point nor *vice versa.*

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[1] R. H. Swendsen, Phys. Rev. Lett. **52**, 2321 (1984).

[2] F. J. Wegner, Phys. Rev. B **5**, 4529 (1972).

[3] Nonlinear scaling fields may be neglected here since they yield weaker singularities than $|\tilde{t}|^{\theta}$; see A. Aharony and M. E. Fisher, Phys. Rev. B **27**, 4394 (1983).

[4] E.g., M. E. Fisher and J.-H. Chen, in *Phase Transitions: Cargese 1980,* edited by M. Levy, J.-C. Le Guillou, and J. Zinn-Justin (Plenum, New York, 1982), p. 169, Sect. 4.

[5] Since the *location* of a fixed point is expected to be nonuniversal, this cannot be excluded *a priori.*

[6] See F. J. Wegner, J. Phys. C **7**, 2098 (1974).