Comprehensive Theory of Flow Properties of ^{3}He Moving through Superfluid ^{4}He in Capillaries

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We propose a new set of hydrodynamic equations for flow of ^{3}He through superfluid ^{4}He. They differ from earlier work by including both a mutual friction force between ^{3}He and ^{4}He and a viscous force. Both the mechanical vacuum model, in which mutual friction is neglected, and an earlier model by the authors, without viscous force, are limiting cases of this new description. An experimental setup is devised in which both forces play a significant role. The results of this experiment are in agreement with the theory.

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One of the most intriguing questions with respect to the hydrodynamics of ^{3}He-^{4}He mixtures is the interrelation between the model in which the mutual friction between ^{3}He and ^{4}He is neglected[1,2] and the model in which the mutual friction plays the dominant role.[3] When mutual friction is absent, the flow properties are determined by the viscous forces on the ^{3}He quasiparticle gas. The validity of this point of view was supported by experimental evidence obtained by Wheatley, Rapp, and Johnson[4] at rather low ^{3}He flow rates (typically 50 μmol s^{-1}). On the other hand, experiments performed at the Eindhoven University of Technology at larger flow rates (typically 500 μmol s^{-1}) showed that the mutual friction force plays the dominant role.[3,5] In the latter experiments it was observed that the pressure changes in tubes in which ^{3}He moves through ^{4}He were negligible (typically smaller than 1 Pa). The new point of view presented in this paper is that both sets of experiments have been performed in regimes where either the viscous force or the mutual friction force dominates the other. In this paper we will show that the results in both flow regimes can be unified in one theoretical description that includes both forces. Results of an experiment in which for the first time the effects of both forces have been measured are reported. The new description connects the ^{3}He-^{4}He hydrodynamics to the theory on superfluid turbulence in pure ^{4}He II.[6] The results are also of practical importance for calculating the properties of ^{3}He-circulating dilution refrigerators, such as the intrinsic minimum temperature.

We consider a flow of ^{3}He through a long tube with length L and with arbitrary uniform cross section of area A, and assume that the total molar flow rate of the ^{4}He component equals zero. In our discussion we use the hydrodynamic equations derived by Khalatnikov,[1] and add a mutual friction force between the normal and superfluid components to the equation of motion for the superfluid component. In our point of view a nonzero relative velocity between ^{3}He and ^{4}He maintains a vortex tangle in the superfluid which leads to this mutual friction force. This idea is based on the similarity with the situation of counterflow in pure ^{4}He II. In the stationary situation and at the low velocities and temperatures under consideration, the energy conservation law in case of adiabatic flow can be written as[7]

$$\dot{Q} + \dot{n}_3(\mu_3 + TS_F) = \text{const}, \tag{1}$$

where $\mu_3 + TS_F$ is the molar ^{3}He osmotic enthalpy and $\dot{n}_3$ the molar ^{3}He flow rate. The heat flow through the fluid is denoted by $\dot{Q}$. Using the Gibbs-Duhem relation in Eq. (1) yields

$$\frac{d\dot{Q}}{dl} + \dot{n}_3\left(T\frac{dS_F}{dl} + V_3\frac{dp}{dl} - \frac{1-x}{x}\frac{d\mu_4}{dl}\right) = 0, \tag{2}$$

where T is the temperature, x the molar ^{3}He concentration, l the distance from the entrance of the tube, and p the pressure. The volume and entropy per mole ^{3}He are denoted by V_3 and S_F, respectively, and μ_4 is the molar ^{4}He chemical potential. A treatment of the thermodynamic properties of liquid ^{3}He-^{4}He mixtures is given in Ref. 7.

From the momentum conservation law follows an equation for the pressure difference: $\nabla p = \eta \nabla^2 \mathbf{v}_3$, where η is the viscosity of the mixture and $\mathbf{v}_3$ the ^{3}He velocity. Similar to the situation of superfluid turbulence in pure ^{4}He there could exist a frictional force between the ^{4}He component and the wall, which would give rise to an excess pressure difference.[6] We will not consider this possibility in this paper. In contrast with Ref. 3 we assume that there is Poiseuille flow and that v_3 equals zero at the wall. It follows that

$$dp/dl = -\eta \dot{n}_3 V_3 Z/L, \tag{3}$$

where Z is a geometrical factor, called the impedance factor.

To the equation of motion for the superfluid component in the steady state we add a mutual-friction-force density $\mathbf{F}_{43}$, caused by the interaction between the ^{3}He and the vortex tangle: $\nabla\mu_4 = -V_4\mathbf{F}_{43}$, where

V_4 is the volume of 1 mole of ^{4}He. In analogy with counterflow in ^{4}He II it is assumed that the mutual friction force is proportional to the cube of the relative velocity of the normal and superfluid components.[8,9] Because of the translational symmetry of the flow channel the velocities are directed along the axis of the tube. Therefore, the chemical potential of the ^{4}He component is uniform in a cross section of the tube and thus the relative velocity is uniform too. The average velocity of the ^{4}He component equals zero in our situation, so that the mutual friction force can be written as

$$d\mu_4/dl = \chi(\dot{n}_3/A)^3. \tag{4}$$

This relation, with the empirical constant $\chi = 1\times 10^{-8}$ kg s m^7 mol^{-4}, is in agreement with our earlier experiments,[3] where a $\dot{n}_3^\alpha$ dependence of $d\mu_4/dl$ was found with $\alpha = 2.8 \pm 0.4$. In pure ^{4}He there is a critical value of the relative velocity between the normal and superfluid components, below which mutual friction does not occur. For the flow channels in consideration the effect of a critical value of the relative velocity has not been observed. Therefore, we will not consider this possibility in this discussion.

For ^{3}He concentrations close to the concentration of the saturated solution at zero temperature, x_0, and temperatures below 50 mK, the following relations[7] hold within 2%: $S_F = C_0 T$, $\eta = \eta_0/T^2$, $\dot{Q} = -A(\kappa_0/T)dT/dl$, and $V_3 = V_{3d}$. The numerical values of the constants C_0, η_0, κ_0, and V_{3d} are given in Ref. 3. Substitution of (3) and (4) in (2) and definition of t, λ, and β by $t = T/T_0$, $\lambda = l/L_0$, and

$$\beta = \frac{\eta_0 V_{3d}^2 C_0 \kappa_0}{[(1-x_0)\chi\kappa_0/x_0]^{3/2}} \frac{A^4 Z}{L\dot{n}_3^3},$$

where

$$T_0 = \left(\frac{1-x_0}{x_0}\frac{\chi\kappa_0}{C_0^2}\right)^{1/4}\left(\frac{\dot{n}_3}{A}\right)^{1/2},$$

and

$$L_0 = \left(\frac{x_0}{1-x_0}\frac{\kappa_0}{\chi}\right)^{1/2}\left(\frac{A}{\dot{n}_3}\right)^2,$$

yields the dimensionless equation for the temperature distribution:

$$-\frac{d}{d\lambda}\left(\frac{1}{t}\frac{dt}{d\lambda}\right) + \frac{1}{2}\frac{dt^2}{d\lambda} - 1 - \frac{\beta}{t^2} = 0. \tag{5}$$

The term β/t^2 gives the ratio of the contributions to the temperature rise due to the viscous force and the mutual friction force, respectively. Equation (5) has been solved analytically by van Haeringen, Staes, and Geurst[10] in both the limiting cases that mutual friction is absent ($\beta = \infty$) and that the viscous force is absent ($\beta = 0$).

For large values of β/t^2, like in the experiment described in Ref. 4 ($\beta/t^2 \approx 2\times 10^3$), the viscous force is dominant. From Ref. 4 and Eq. (3) follows a value for the pressure difference across the flow tube Δp, on the order of 20 Pa, which is easily measured. For small values of β/t^2 like in the experiments performed by Castelijns *et al.*,[3] with β/t^2 on the order of 4×10^{-3}, the viscous force is negligible. Solution of Eq. (5) yields a large temperature rise in the impedance. Since the viscosity decreases strongly with temperature, Eq. (3) gives a pressure drop over the flow channel on the order of 1 Pa. This is difficult to measure accurately, because hydrostatic pressure differences due to density variations in the liquid are on the same order of magnitude. In the range where the value of β/t^2 is on the order of 1 both forces play an equally important role.

In order to test the theory in the intermediate range we devised a flow impedance consisting of 28 cylindrical tubes in parallel, each of diameter 0.28 mm and length 23 mm, installed at the dilute-exit tube of a mixing chamber as described in Ref. 3. For ^{3}He flow rates in the range of 0.15 to 0.5 mmol s^{-1}, β ranges from 500 to 15 and T_0 ranges between 2.3 and 4.3 mK. The mixing chamber temperature T_m varied between 30 and 60 mK. On the basis of Eq. (3) it is expected that for this flow impedance the pressure difference will be well above 1 Pa. Since a difference in the chemical potential of the ^{4}He component is also detectable, it should be possible to demonstrate the effects of both the viscous and the mutual friction forces

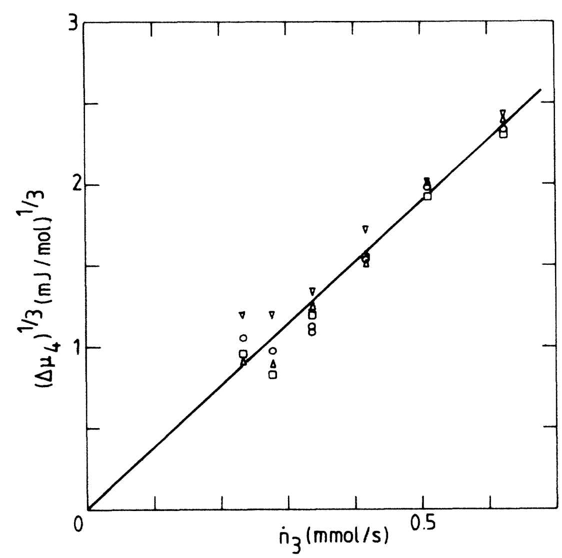

FIG. 1. Cube root of the difference in ^{4}He chemical potential over the flow impedance as a function of the molar ^{3}He flow rate. The symbols correspond to different values of T_m (squares, 30 mK; circles, 40 mK; triangles, 50 mK; inverted triangles, 60 mK). The line obeys $\Delta\mu_4 = 1.2\times 10^{-8}(\dot{n}_3/A)^3 L$.

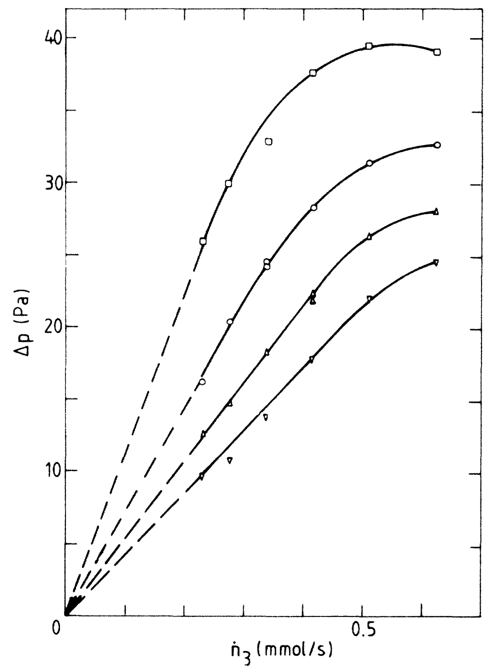

FIG. 2. Pressure differences Δp over the flow impedance as functions of the molar ^3He flow rate for several values of T_m: squares, 30 mK; circles, 40 mK; triangles, 50 mK; inverted triangles, 60 mK. The lines are for visual aid only. The viscosity parameter η_0 is determined from the dashed parts of the curves.

in one experiment. The pressure differences Δp were measured as functions of $\dot{n}_3$ for several values of T_m. From the measured temperatures and ^3He concentrations at both ends of the impedance, μ_4 is calculated. In Fig. 1, the cube root of the difference in ^4He chemical potential between the exit and the entrance of the impedance, $\Delta\mu_4$, is plotted as a function of $\dot{n}_3$. It can be seen that mutual friction is present ($\Delta\mu_4 \neq 0$) and satisfies Eq. (4). In Fig. 2 the measured pressure differences for various values of T_m are plotted as functions of the ^3He flow rate. From the linear part of these graphs the values of the parameter η_0 is calculated. It varied from $(5\pm1)\times10^{-8}$ Pa s K^2 at $T=30$ mK to $(6.5\pm1)\times10^{-8}$ Pa s K^2 at $T=60$ mK. These values are in good agreement with the measurements of Kuenhold, Crum, and Sarwinski,[11] who found a value for η_0 increasing with temperature, with a low-temperature limit of 5×10^{-8} Pa s K^2. For higher flow rates the increasing temperature in the tube leads to a decreasing viscosity of the mixture. This effects results in a concave Δp-$\dot{n}_3$ relationship.

In conclusion, we set up a new theory to describe the hydrodynamics in ^3He-^4He mixtures, which includes both a mutual friction force, presumably caused by the interaction between the ^3He and a ^4He vortex tangle, and a viscous force. This theory is in agreement with experiments, in which one force dominates the other, as well as with our measurements in a regime where both forces are equally important. With this description one of the main problems in ^3He-^4He hydrodynamics at low temperatures is solved. Presently we are investigating the effects of an eventual critical velocity and friction between ^4He and the walls, which were neglected in this paper, and the microscopic derivation of Eq. (4).

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