Superconducting-Normal Phase Boundary of a Fractal Network in a Magnetic Field

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Experimental measurements of the superconducting-to-normal phase boundary for a Sierpenski-gasket wire network are presented. The fractal phase-boundary curve, $T_c(H)$, shows four orders of dilational invariance and is in excellent quantitative agreement with theoretical predictions. The fracton dimension, $\bar{\bar{d}}$, has also been measured and is found to be 1.35 ± 0.02, consistent with the calculated value of 1.365 for the planar Sierpinski gasket.

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In recent years considerable attention has been devoted to the study of the properties of disordered systems with the hope of a better understanding of the percolative aspects of phenomena such as the spin-glass,[1] metal-insulator, and superconducting[2] transitions. Central to many of these studies are the concepts of randomness and frustration. A particularly elegant experimental model system for isolating the effects of frustration is the regular array of superconducting wires[3,4] (or Josephson junctions[5]) in which the level of frustration is continuously tunable via an external magnetic field. It is essential to incorporate randomness in order more closely to model real systems (glasses, percolating films, etc.). Recently, it has been proposed that a family of regular, fractal networks (in two dimensions, the Sierpinski gasket) share important geometrical features with the backbone of the infinite cluster of the percolation problem.[6] In addition, because of their dilational symmetry, statistical mechanical and transport problems are exactly solvable on these fractals, making them counterparts of the regular arrays and attractive candidates as model systems.[7] Finally, study of the Sierpinski gasket (SG) network is inherently interesting because of its lack of translational invariance and its anomalous (fractal and fracton) dimensionalities.[6,8]

In this Letter we report our experimental investigations of the properties of a superconducting Sierpinski-gasket network in a magnetic field. This is an exactly solvable model problem incorporating an adjustable level of frustration and important geometrical elements of randomness. We have found that the richly structured phase boundary, $T_c(H)$, of the gaskets is quantatively fitted by theoretical predictions,[9] and that the self-similarity of the gasket manifests itself in up to four orders of dilational invariance in our experimentally determined phase boundary. Further, from a quantitative analysis of the depression of T_c in a magnetic field we have made the first experimental determination of the critical exponent for diffusion,[10] Θ (and thus the fracton dimension), and find it to be in agreement with calculations.

The samples were prepared by evaporation of pure aluminum films onto oxidized silicon substrates. These films, 100 nm thick, were deposited directly onto the substrate through a lift-off mask written into a multilayer resist by use of a Cambridge EBMF-2-150 Electron Beam Microfabricator. The gaskets (details are shown in Fig. 1) are of tenth order and have linewidths of 0.3–0.4 μm. The elementary triangles which generate the network have an area of $a = 1.38 \pm 0.01\ \mu\text{m}^2$ as determined from micrographs. Because of restrictions in the pattern-generating system the triangles are not truly equilateral; rather they are are isosceles with both the base and height equal to 1.66 μm.

The normal-state sample resistance, approximately 20 Ω at 4 K, was measured with a four-probe ac bridge biased at about 1 μA. The important intrinsic properties such as the resistivity, $\rho = 0.42\ \mu\Omega$ cm, and the

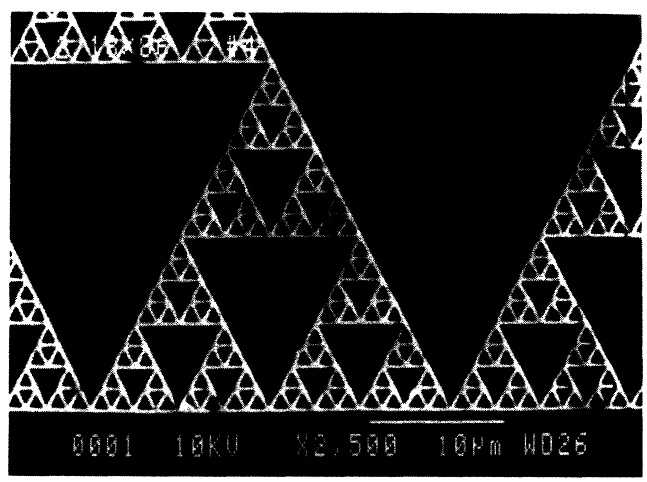

FIG. 1. An electron micrograph of a fourth-order section of the Sierpinski gasket. The size bar represents a length of 10 μm.

coherence length, $\xi(0)=0.26$ μm, were inferred from resistance measurements on coevaporated 2D films and from the quadratic background of the phase boundary, $T_c(H)$,[3,11] which is associated with the one-dimensional character of the wires of the network. Temperature was determined from the resistance of a calibrated Ge resistor.

In order to generate the T_c-H phase boundary the sample resistance itself was used as the sensor in the temperature-controlling system. Typically, the temperature was regulated so as to fix the sample resistance at one-half of its normal-state value, R_N. The superconducting phase-boundary curves, $T_c(H)$, were then swept out by stepping of the magnetic field in small (millioersted) increments and averaging of the transition temperature at each value of the field. Temperature measurements were reproducible from sweep to sweep and had a relative accuracy of about 50 μK.

A dramatic series of oscillations of T_c with field is observed on top of the quadratic background (Fig. 2). This complex behavior, reminiscent of the Little-Parks effect in cylinders[12] and the phase boundaries seen in regular arrays,[3] is due to the highly ramified structure of the SG network. The fundamental period $H_0=15$ Oe (the first period runs from -7.5 to 7.5 Oe) is due to the fluxoid-quantization condition on the elementary triangles and corresponds to an area of $\phi_0/H_0=1.38$ μm^2 which agrees with the value measured directly from micrographs. Here $\phi_0 \equiv hc/2e$ is the superconducting flux quantum. We have observed up to five orders of fundamental oscillations on either side of the field origin. Samples which appeared to be regular under the scanning electron microscope had phase

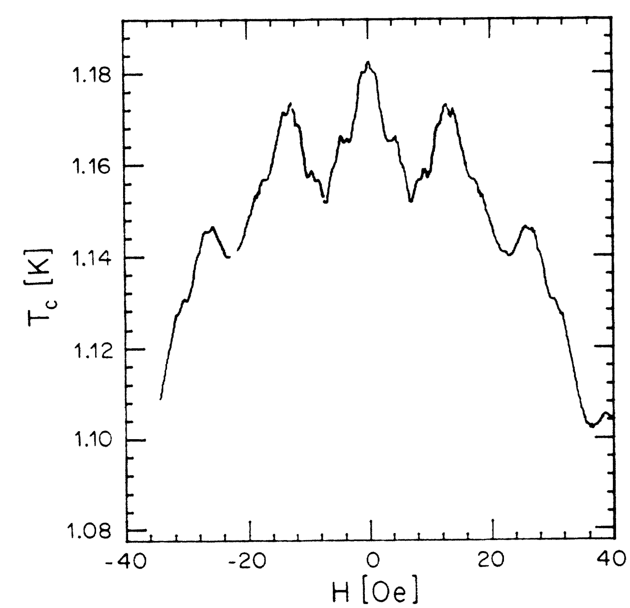

FIG. 2. The superconducting-normal phase boundary, $T_c(H)$.

boundaries consistent with theory. Those with geometrical errors did not exhibit the full range of structure.

The nature of the superconducting states on the SG in a magnetic field has recently been the object of a great deal of theoretical work.[13–15] Near the second-order phase boundary, where $w \ll \lambda, \xi$ in our samples, the properties of the gasket are expected to be well described by the linearized form of the Ginsburg-Landau equation, which is formally identical to the Schrödinger equation for a doubly charged particle in a magnetic field.[4] Alexander[13,14] and de Gennes[15] have expressed the network Ginsburg-Landau equation as an eigenvalue problem in terms of the order parameter at the nodes, Δ_α. For a node α connected to neighboring nodes β, with wire lengths L_0, the equations are written

$$\epsilon\Delta_\alpha \equiv [z\cos(L_0/\xi)]\Delta_\alpha = \textstyle\sum_\beta \Delta_\beta \exp[-i\gamma_{\alpha\beta}]. \quad (1)$$

Here z is the node coordination number (4 in the case of the planar SG) and $\gamma_{\alpha\beta}=(2\pi/\phi_0)\int_\alpha^\beta \mathbf{A}\cdot d\mathbf{l}$ is the circulation of the vector potential along the link $\alpha\beta$. For low-order gaskets (zero order is a single triangle, first order is built from three triangles, etc.) these questions may be solved directly. Rammal and Toulouse[9] have calculated the spectrum for a second-order gasket in a magnetic field (the phase boundary is the band edge of this spectrum); however, for higher-order structures this method becomes hopelessly difficult. Recursion relations exist[9] which allow one to determine, in principle, the eigenvalue spectrum of (1) at arbitrarily high orders.

In Fig. 3(a) we have plotted the first period of the experimentally determined phase boundary (solid line). The applied field is scaled in units of the superconducting flux quanta per elementary triangle: $\phi/\phi_0=H(a/\phi_0)$. The points overlying the data represent the band edge of the second-order spectrum calculated by Rammal and Toulouse. The temperature axis has been scaled to fit the data at $\phi/\phi_0=\frac{1}{2}$. Experimentally, the depression of the transition temperature at this point is $\Delta T_c(\phi/\phi_0=\frac{1}{2})/T_{co}\equiv 0.025$. This is in excellent agreement with the theoretical prediction

$$\frac{\Delta T_c}{T_{co}}(\phi/\phi_0=\tfrac{1}{2}) = \frac{\xi^2(0)}{L_0^2}\arccos^2\left[\frac{\epsilon(\phi/\phi_0=\frac{1}{2})}{z}\right] = 0.024. \quad (2)$$

We have used the mean length $L_0=1.73$ μm and $\xi(0)=0.26$ μm as discussed earlier. In addition to the good agreement relating to the magnitude of the dip in T_c, the match of the functional form of the theory to the data is also quite good. A detailed inspection of the data, however, reveals a great deal of fine structure

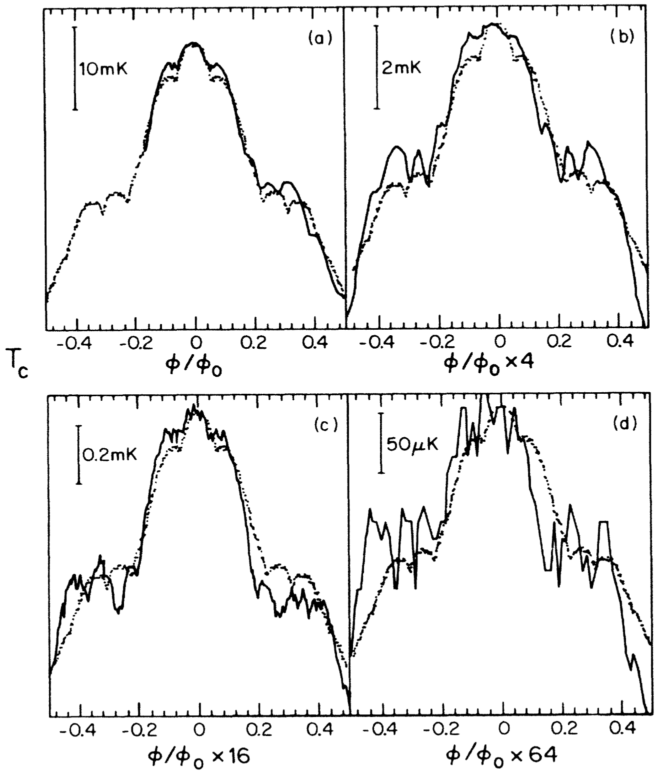

FIG. 3. (a) The superconducting transition temperature as a function of the normalized flux through an elementary triangle of the gasket. The solid line is experimental data and the points represent the theory for a second-order gasket scaled to fit the data at $\phi/\phi_0 = \frac{1}{2}$ (Ref. 9). In (b)–(d) the field axes have been expanded 4, 16, and 64 times, respectively, in order to highlight the small-field structure in $T_c(H)$. The self-similar nature of the curve is made apparent by our superimposing the second-order theory on each plot.

which presumably would be described by the higher-order theory. The second-order curve provides an envelope for all higher-order calculations of the band edge. The fine detail at low fields is in marked contrast to the linear behavior in regular arrays[3,5] and the quadratic behavior in the Little-Parks effect.[12]

The apparent difficulty relating to the lack of theory at small fields may be resolved by use of the dilational symmetry of the gasket and thus the phase boundary curve itself. From one order of the gasket to the next there is a factor of 4 decrease in the enclosed area, corresponding to a factor of 4 dilation in the flux. In Fig. 3(b) we have expanded the field axis of our data by a factor of 4 in order to highlight the structure in T_c at small fields. The striking similarity to the original data [Fig. 3(a)] is made clear by overlaying the (also expanded) second-order theory. Two more orders of this self-similarity are demonstrated in Figs. 3(c) and 3(d) before the temperature resolution limit of about 50 μK is reached. The smallest field at which we see resolvable structure ($64\times\phi/\phi_0 \approx 0.15$ or $H \approx 30$ mOe) corresponds to a length scale of about $[\phi_0/H]^{1/2} = 26\ \mu$m, or an area which includes about 100 elementary triangles. In comparison, structure observed in the regular arrays was due to, at most, groups of about 5 elementary cells.[3]

In systems of integral dimension the exponents governing thermodynamic properties [e.g., the density of states: $N(E) \sim E^{(d/2)-1}$] are well known and depend on the Euclidean dimension d. It is tempting to assume that similar relationships hold on the fractal lattice with the substitution of the fractal dimension, D, for d. Alexander and Orbach[10] have shown, however, that for such properties d must be replaced by yet another anomalous dimension, the so-called fracton dimensionality, $\bar{\bar{d}} = D/(1+\frac{1}{2}\Theta)$, where Θ is the anomalous diffusion exponent.[10,16,17] For the planar SG, $D = \ln 3/\ln 2$ and $\Theta = (\ln 5/\ln 2) - 2 = 0.322$. Rammal and Toulouse[9] point out that in the limit of small fields the phase boundary should obey a power law $T_{co} - T_c \propto H^{4/(4-\Theta)}$. In the regular array, where Θ vanishes, one recovers the linear relationship between ΔT_c and H. The small-field condition, roughly $H < \phi_0/L^2 \simeq 5\times10^{-6}$ Oe, is far beyond our resolution. L is the size of the entire gasket. Using "nesting property II" from Rammal and Toulouse, however, we note that the eigenvalues of the nth-order gasket at the fields $H_n = \phi_0/(2\times 4^n a)$ remain in the spectrum at all higher orders. Specifically, the band-edge solutions at these fields are of the form

$$\Delta T_c(H_n) \equiv T_{co} - T_c(H_n) \simeq H_n^2 L_n^{2-\Theta}, \tag{3}$$

where $L_n = 2^n L_0$ is the size of the nth-order gasket. Noting that $L_n^2 = (a\phi_0/2H_n)L_0^2$ it follows from (3) that $\ln\Delta T_c(n) = (1+\Theta/2)\ln H_n + \text{const}$. Thus, we expect the points $\Delta T_c(\phi_n)$ to fall on a single curve defined by $T_{co} - T_c \propto H_n^{1+\Theta/2}$ with $1+\Theta/2 = 1.161$.

In Fig. 4 we have plotted $T_{co} - T_c$ vs H on logarithmic axes, with arrows indicating the field values where $\phi/\phi_0 = \frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \ldots$ The $H^{1+\Theta/2}$ behavior is represented by the solid line. A least-squares analysis of the data gives $(1+\Theta/2) = 1.17 \pm 0.02$. Thus, we determine the fracton dimensionality $\bar{\bar{d}} = D/(1+\Theta/2) = 1.35 \pm 0.02$, consistent with the theoretical value $\bar{\bar{d}} = 1.365$.[18]

Before concluding we mention an unexplained feature of our data. Above we have used the value $H_0 = 15.0$ Oe for the period corresponding to one flux quantum per elementary triangle. This is the value of the field separating adjacent minima in the phase boundary and is consistent with the measured area of a triangle, $a = 1.38\ \mu\text{m}^2$. Inspection of the data in Fig. 2, however, reveals that adjacent *maxima* in $T_c(H)$ are separated by $H_0 = 13.0$ Oe. This discrepancy in the periodicity deserves further study (see Fig. 1).

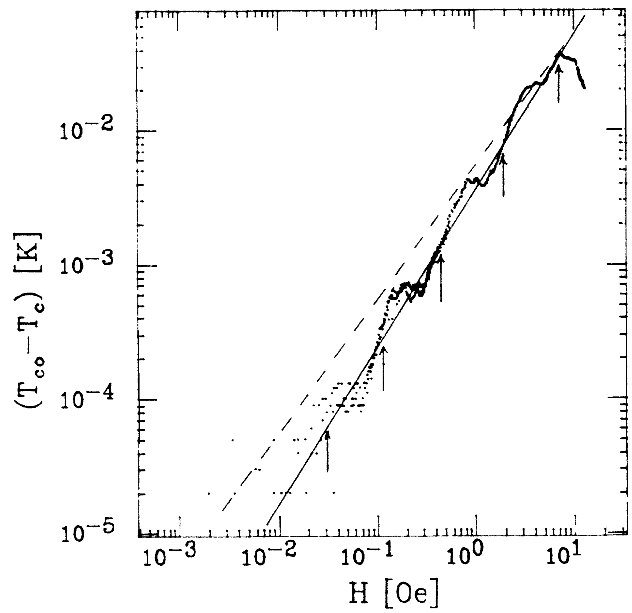

FIG. 4. The suppression of the superconducting transition temperature as a function of magnetic field, on logarithmic axes. The arrows mark the field values corresponding to $\phi/\phi_0=\frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \frac{1}{128}$, and $\frac{1}{512}$. As discussed in the text, the solid line through these points has a slope $1+\Theta/2=1.17$, consistent with the calculated value of 1.161. The dashed line has a slope of 1.0 for reference.

In summary, we have made measurements of the superconducting-normal phase boundary of a high-order Sierpenski gasket constructed from submicrometer-width Al wires. The phase boundary is found to exhibit structure over an exceptionally wide range of fields, satisfying $\frac{1}{500}<\phi/\phi_0<5$. Further, it is shown to be self-similar over four orders of dilation and is in good quantitative agreement with theory. Finally, the critical exponent for diffusion, Θ (and thus the fracton dimensionality), has been experimentally determined and is consistent with theoretical values.

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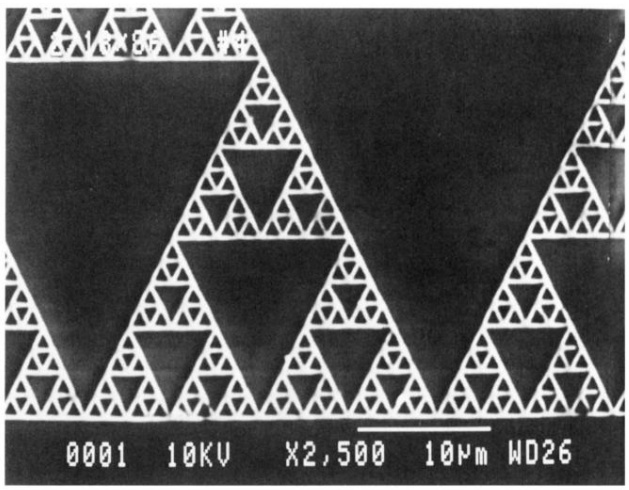

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