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Dynamic Transition in a Hierarchical Ising System

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A one-dimensional kinetic Ising model, with hierarchical couplings, is solved. We find algebraic relaxation of the magnetization, with a temperature-dependent exponent, and breakdown of dynamic scaling. The nonlinear relaxation time diverges with the system size below a dynamic transition temperature. Possible relevance to dilute Ising systems at percolation is discussed.

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One of the most important factors which determine the time relaxation of random systems (amorphous or glassy systems, spin-glasses, dilute alloys, etc.) is the existence of a very wide range of characteristic time scales.[1,2] Such hierarchies of time scales also arise in molecular diffusion on complex macromolecules,[3] computing structures,[4] etc. This motivated several recent studies of random walks on systems with a hierarchical set of hopping rates (or barrier heights).[5-10] These models exhibit a very interesting sharp dynamic transition: If the ratio between consecutive hopping rates in the hierarchy is R then the diffusion is "normal" for $R_c < R$, and "anomalous" for $0 < R < R_c$, where the diffusion coefficient D (or the conductivity) vanishes.[6,11]

Random walks of diffusing domain walls are believed[12] to control the low-temperature dynamics[13,14] of Ising spin systems whose equilibrium magnetization is zero at finite temperatures. This is the case on some ramified fractal structures, such as the incipient infinite cluster at the percolation threshold[15,16] or model fractals (e.g., the Sierpinski gasket[15]). In such systems, the maximum energy barrier that a wall must cross increases with their size, L.

This led several groups[15,17] to predict that the relaxation time τ diverges as $\tau \sim L^{z(T)}$ for $L \ll \xi$ and $\tau \sim \xi^{z(T)}$ for $L \gg \xi$, where ξ is the static correlation length ($\xi \to \infty$ as $T \to 0$), and $z(T) \sim 1/T$. This represents a breakdown of conventional dynamic scaling,[18] for which the dynamic exponent z must be temperature independent.

In fact, the fractal geometry implies a whole hierarchy of energy barriers, and hence of hopping rates which have temperature-dependent ratios, of the form $\ln R \sim 1/T$. In view of the above quoted results on hierarchical models, one might therefore expect a similar dynamical transition, below which some appropriately defined diffusion coefficient vanishes. However, as the temperature is raised, one is no longer *a priori* justified in using the single-domain-wall diffusion argument. Instead, one should consider the full master equation for the spin distribution function $P(\{s_i\};t)$.[13,14,19] Surprisingly, recent exact treatments of this full equation claimed to recover dynamic scaling at *all* temperatures.[20] Dynamic scaling was also used in the analysis of the pioneering experiments on dilute magnets,[21] and in other recent publications.[22] This Letter aims at resolving this controversy, and to find whether a sharp dynamic transition[6] also occurs for Ising spin dynamics.

To this end we consider a simple one-dimensional kinetic Ising model, which contains the hierarchical features mentioned above. We solve the full Glauber equation, and find the dependence of the system's magnetization M on time t, on L, and on T. Although the model is simple, the two above-mentioned approaches do yield different results for it; our solution resolves the controversy, in a manner that may be relevant for more complex situations as well.

For $L \to \infty$ and long times, we find *algebraic relaxation*, $M \sim t^{-x(T)}$ with $x(T) \sim T$. This implies that the nonlinear relaxation time, $\tau_{\rm nl} = \int M(t)\,dt$, is infinite for $T < T_c$, where $x(T_c) = 1$. This novel *dynamic transition* at T_c resembles that of the random-walk

problem, in which $D^{-1} \to \infty$. For finite L we find

$$\tau_{nl} = \tau_0(T) L^{-1+1/x(T)}, \quad T < T_c, \tag{1a}$$

$$\tau_{nl} = \tau_0(T) \ln L, \quad T = T_c, \tag{1b}$$

$$\tau_{nl} = \frac{\tau_0(T)(1 - L^{-1+1/x(T)})}{2 - 2^{1/x(T)}}, \quad T > T_c \tag{1c}$$

for all L. Our results imply *breakdown of dynamic scaling.*

We consider a ferromagnetic Ising chain of length L, with nearest-neighbor couplings assigned in a hierarchical manner (see Fig. 1). Odd bonds have strength $K_1 = J_1/kT$. Couplings of strength $K_2 + mK_3$ are assigned, in the hierarchical manner indicated, to fractions $2^{-(m+2)}$ of bonds. We use a renormalization-group procedure,[8] decimating the spins indicated by σ in Fig. 1. The spins μ that survive the decimation form a linear chain with the same hierarchical structure. If we set $u_i = \cosh K_i$ and $v_i = \tanh K_i$, the static recursion relations are[23]

$$v_1' = v_1^2 v_2, \quad K_2' = K_2 + K_3, \quad K_3' = K_3. \tag{2}$$

ξ is determined by the (weakest) K_1 bonds only: $\xi \sim e^{2K_1}$ for $K_2 > K_1 >> 1$. The master equation for $P(\{s\};t) = P(\{\sigma\},\{\mu\};t)$ takes the form[13]

$$\frac{d}{dt} P(\{s\};t) = -\sum_i (1 - p_i) W_i(s_i) P(\{s\};t), \tag{3}$$

where p_i is a flip operator for spin i,

$$p_i f(s_1, s_2, \ldots, s_i, \ldots, s_L) = f(s_1, s_2, \ldots, -s_i, \ldots, s_L).$$

We chose transition rates $W(s_i \to -s_i) = W_i(s_i)$ given by

$$W_i(s_i) = \exp\{-s_i[K^{(i-1,i)} s_{i-1} + K^{(i,i+1)} s_{i+1}]\} e^{-|\Delta K^{(i)}|},$$

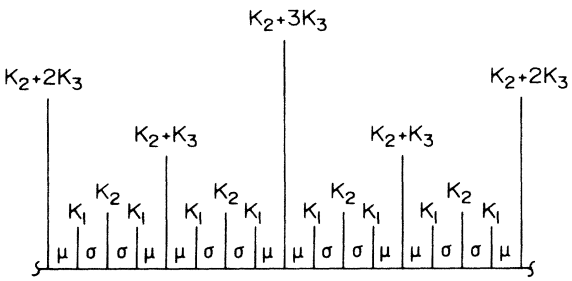

FIG. 1. Part of an infinite Ising chain. The vertical lines indicate the strength of the nearest-neighbor, hierarchically assigned bonds K_1, K_2, $K_2 + mK_3$ between spins (σ and μ). Decimation of $\{\sigma\}$ yields a chain whose couplings are also hierarchical.

where $\Delta K^{(i)} = K^{(i-1,i)} - K^{(i,i+1)}$; $K^{(i,i+1)}$ denotes the coupling of s_i to s_{i+1}. Our $W(s_i)$ satisfies detailed balance and is chosen so that the rate associated with moving a domain wall a single step in a direction that lowers the energy is set equal to unity.

Following standard procedure,[20] we represent $P(\{s\};t)$ as $P_e(\{s\})\phi(\{s\};t)$, where $P_e(\{s\}) = \exp[-\mathcal{H}(\{s\})]/Z$ (Z is the partition sum). We take $\phi(\{s\};t) = 1 + h(t)\sum_i s_i$, keeping only the slowest operator that forms an invariant subspace.[20] The manner in which $h(t)$ scales under decimation yields information about scaling of the slowest mode of the system. After decimation, the master equation for $P'(\{\mu\};t') = \mathrm{Tr}_{\{\sigma\}} P(\{\sigma\},\{\mu\};t)$ becomes

$$\frac{d}{dt'} P'(\{\mu\};t') = -\sum_{i'} (1 - p_{i'}) W_{i'}'(\mu_{i'}) P'(\{\mu\};t'). \tag{4}$$

Again $P' = P_e' \phi'$; P_e' and W' are expressed in terms of the renormalized static Hamiltonian (2) precisely as above, and ϕ' is linear in $\sum_i \mu_i$.[23] Time gets rescaled, $t' = t/\Lambda$, where

$$\Lambda = \left[1 + \frac{v_1(1+v_2)}{1+v_1^2 v_2}\right] \frac{u_1}{u_1'} e^{K_1' - K_1}. \tag{5}$$

In order to highlight the manner in which our treatment deviates, from this point on, from standard dynamic renormalization-group procedures,[20] we first restate the basic premises of the latter approaches. To calculate the relaxation time, one evaluates $\Lambda = b^z$ from (5) at a fixed point, with b the length rescaling factor ($b = 2$ in our case). The renormalization group is now iterated n times, until the couplings reach a high-temperature regime $K^{(n)}$, for which the relaxation time $\tau^{(n)} = \tau(K^{(n)})$ is some finite constant. Since $\tau = \tau(K^{(0)}) = \Lambda^n \tau^{(n)}$, and $\xi = \xi(K^{(0)}) = b^n \xi^{(n)}$, one finds $\tau \sim \xi^z$. Thus the magnetization associated with sites that survived n interations is found to be given by $M^{(n)} \sim \exp[-t/\tau]$. Since for translationally invariant systems all sites are equivalent, the total magnetization is also given by precisely the same form.

In hierarchical systems, however, the situation differs in two important ways. *First,* in our case (for $L \to \infty$), no matter how many iterations are taken, the surviving spins still constitute a hierarchical system. For sufficiently large n, our system maps onto a problem with $K_1^{(n)} << 1$, i.e., nearly decoupled pairs of spins (see Fig. 1); the coupling within these surviving pairs is, however, arbitrarily large, $K_2^{(n)} + mK_3$! Therefore, even for large n one cannot assume $\tau^{(n)} = \text{const}$, and identify Λ with b^z. Since our Glauber model is one of single-spin-flip dynamics,[24] the strong

interpair bonds must also be broken for equilibration, and the hierarchical nature of the *dynamic* problem is present after any number of iterations. This does *not* hold for statics: For $K_1^{(n)} \ll 1$ the correlation length is ~ 1, and therefore we do get $\xi \sim b^n$.

The *second* important distinctive feature of our hierarchical model stems from the fact that since the problem is not transitionally invariant, not all spins relax at the same rate. Therefore at each step of decimation we must evaluate and add up the contribution to the total magnetization that is due to the spins that are being eliminated. The exact decimation of Achiam[20] stops at calculating Λ, while Henley's heuristic argument[15] does take the hierarchical nature of the problem into account. Stopping after the decimation,[20] obtaining Λ, and using the standard arguments yields dynamic scalings. Calculation of M changes this conclusion, and resolves the controversy.[15,20]

So far our decimation was exact. In what follows we consider the model in the regime $K_2 > K_1 \gg 1$, and apply the standard approximations of using the "linearized recursion relations," up to the matching point $K_1^{(n)} \ll 1$;

$$\Lambda \approx 2; \quad \exp[2K_1^{(m)}] \simeq 2^{-m}\exp[2K_1]; \qquad 2^n \simeq e^{2K_1} \approx \xi. \tag{6}$$

The magnetization per site is given by $M(t) = (M_\mu + M_\sigma)/2$, where

$$M_\mu = (2/L)\mathrm{Tr}_{\{\mu\}}\mathrm{Tr}_{\{\sigma\}}P(\{\sigma\},\{\mu\};\tau)\sum\mu_i, \tag{7a}$$

$$M_\sigma = (2/L)\mathrm{Tr}_{\{\sigma\}}\mathrm{Tr}_{\{\mu\}}P(\{\sigma\},\{\mu\},t)\sum\sigma_i. \tag{7b}$$

M_μ equals by definition the magnetization per site of the hierarchical model obtained after decimation of the spins $\{\sigma\}$, with couplings given by (2) and time rescaled by Λ; $M_\mu = M^{(1)}(t/\Lambda)$. On the other hand, M_σ is the magnetization per site of a system obtained after decimation of the spins $\{\mu\}$. This yields a linear chain $\{\sigma\}$ with alternating couplings, which within our approximations (6) are given by $K_1' = K_1^{(1)}$ and K_2, yielding $M_\sigma \simeq \exp[-t/\Lambda\tau^{(1)}]$, where $\tau^{(1)} = \exp[2 \times (K_1^{(1)} + K_2)]$. Repeating this procedure n times for M_μ we find

$$M(t) = 2^{-n}M^{(n)}\frac{t}{\Lambda^n} + \sum_{m=1}^{n} 2^{-m}\exp\left(\frac{-t}{\Lambda^m\exp[2(K_1^{(m)} + K_2^{(m-1)})]}\right). \tag{8a}$$

$M^{(n)}$ is the magnetization of very weakly ($K_1^{(n)} \ll 1$) coupled pairs, with bonds $K_2^{(n)} + mK_3$ within the pairs. Using (6) and some algebra one finds[23,25]

$$M(t) = \sum_{m=0}^{\infty} 2^{-(m+1)}e^{-t/\tau_m}, \tag{8b}$$

with $\tau_m = \exp\{2[K_1 + K_2 + mK_3]\}$. As expected, (8b) does not depend on the matching point n. Calculating $M(t)$ by the method of steepest descent, we find ($\tau_0 = e^{2(K_1+K_2)}$)

$$M(t) \simeq (t/\tau_0)^{-x(K_3)}; \quad x(K_3) = (\ln 2)/2K_3 \tag{9}$$

for times $t/\tau_0 > \tilde{t}_1 = x(K_3)$; $\tilde{t}_1$ is obtained by the requirement that the value m_0 that dominates the sum (8) be larger than zero.

Integrating (8b) yields

$$\tau_{\mathrm{nl}} = \xi^2 \sum_m 2^{-(m+1)} W_m^{-1}, \tag{10}$$

where $W_m^{-1} = \exp[2(K_2 + mK_3 - K_1)]$ can be interpreted [see W following (3)] as the inverse rate at which a single domain wall moves over a bond of strength $K_2 + mK_3$; the prefactor $2^{-(m+1)}$ represents the concentration of such barriers. This identifies τ_{nl} as the time it takes a wall to to diffuse a distance ξ. The corresponding diffusion constant,

$$1/D = \tau_{\mathrm{nl}}/\xi^2 = \langle W^{-1}\rangle, \tag{11}$$

is similar to that found in the diffusion problem[6]; D vanishes, and τ_{nl} is infinite, when the temperature-dependent parameter $K_3 = J_3/kT$ is above its dynamic transition value $K_3^c = J_3/kT_c$, where $x(K_3^c) = 1$. As $T \to T_c^+$, τ_{nl} diverges as

$$\tau_{\mathrm{nl}} = \tau_0(2 - e^{2K_3})^{-1}. \tag{12}$$

This concludes our proof, for $L \to \infty$, of the existence of a dynamic transition, in the parameter range $K_2 > K_1 \gg 1$ and arbitrary K_3. Another regime in which the problem is trivially solvable[23,25] is when $K_1 \ll 1$, where Eq. (8b) is self-evident. This strongly supports the existence of a transition at $K_3 = K_3^c$ for all K_1 and K_2.

For *finite* L, the sum in (8) must be cut off at $m_{\max} = \log_2 L$. Therefore, the algebraic behavior (9) is valid only when $0 < m_0 < m_{\max}$, i.e., $\tilde{t}_1 < t/\tau_0 < \tilde{t}_2 = x(K_3)L^{1/x(K_3)}$. For $t/\tau_0 > \tilde{t}_2$, the t dependence of M is dominated by the slowest term, $\exp(-t/\tau_{m_{\max}})$. The sum (10) now yields Eq. (1).

Are our results relevant to dilute Ising systems at their percolation threshold p_c? At p_c the backbone of the infinite incipient cluster is described by the "links and blobs" model,[26] in which two sites at a linear distance L are connected via $L_1(L) \sim L^{1/\nu}$ singly connected bonds ($\xi_p \sim |p - p_c|^{-\nu}$) with exchange constant $K_1 = J/kT$ each, and a hierarchy of blobs of multiconnected spins with linear sizes (l) distributed as[26]

l^{-a}. Each "blob" represents a minimal energy barrier[15,16] $\Delta E(l) \sim 2J(A \ln l + B)$ and a corresponding relaxation time. Ignoring dangling ends, one might renormalize each "blob" into a single bond, with exchange constant $K(l) = K(A \ln l + B)$ and with weight l^{-a}. This reduces to a random version of our model, with K_1, K_2, and K_3 of order $K = J/kT$ and $m \sim \ln l$.

In spite of these similarities with our model, there exist several possibly important differences. First, a detailed application of our results to percolation requires inclusion of dangling ends, averaging over finite clusters, and (possibly) treating the dynamics within blobs in more detail. Another difference concerns the spatially random assignment of the hierarchically distributed effective bonds. Since randomness[11] turned out to be irrelevant in the hierarchical diffusion problem,[6] our present results may also remain valid in the random case.

A crucial difference arises from the fact that in a percolation cluster with $L > \xi$, regions of size ξ are believed[15,17] to contain only barriers smaller than $K_2 + K_3 \ln\xi$. Since equilibration involves only such regions, the sum in (8) should be cut off at $m'_{\max} \sim \ln\xi$. In our model, however, a finite fraction of these regions will contain barriers of size $\ln L$. To describe percolation clusters with $L > \xi$ one should thus use Eqs. (1), with L replaced by ξ. Indeed, at low temperature, our Eq. (1a) becomes $\tau_{\rm nl} = \tau_0(T) \times \xi^{-1+1/x}$, in agreement with the heuristic arguments[15,17] and with numerical simulation.[17,27] However, our paper contains many additional predictions at higher temperatures. In particular, an algebraic decay of the magnetization is expected for intermediate times $\tilde{t}_1 < t/\tau_0 < x(K_3)\xi^{1/x(K_3)}$. Since similar behavior is expected for the spin-relaxation function, fitting its Fourier transform to a Lorentzian for all frequencies[21] may be misleading.

For $L < \xi$, the L dependence of our results should remain valid. This will always happen at sufficiently low T. In computer simulations, with relatively small L, this should apply even at moderate temperatures.

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