Line Functionals and String Field Theory

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An alternative covariant formulation of string field dynamics is given, in which functions of one-dimensional curves (line functionals) are the essential ingredients. Guided by the geometrical principle of manifest reparametrization invariance and the correspondence to Nambu-Goto string dynamics in the $x^0=\tau$ gauge, we derive generalized Dirac-string field equations linear in the momentum density. It is shown that in three dimensions the equations for closed strings admit an equally spaced mass-squared spectrum, including massless particles of spin 0 and 1.

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During the past year, considerable progress has been made towards the incorporation of gauge invariances in the covariant formulation of string field theories. The gauge-invariant, covariant treatment of free bosonic strings and superstrings has been accomplished,[1,2] and the inclusion of interactions appears near at hand.[3] There is every indication that these theories are consistent with older formulations based on the light-cone gauge, even to the point of reproducing the Veneziano scattering amplitudes. More remarkably, they provide a natural explanation for the origin of non-Abelian gauge symmetries and general coordinate invariance.[4]

In several respects, however, the current proposals for covariant string field actions seem less than satisfactory. At the most fundamental level they ignore the basic fact that objects appropriate to string field theory are functions of one-dimensional open or closed curves in space-time, or line functionals Φ[line].[5] The line functional Φ[line] may be written as $\Phi[x^\mu(\sigma)]$, where each line configuration is represented by its coordinates $x^\mu(\sigma)$, $0\leqslant\sigma\leqslant 1$. The fact that Φ is a function of lines means that Φ does not depend on the parametrization of each line, or in other words,

$$x'(\sigma)\cdot p(\sigma)\Phi[x^\mu(\sigma)]=0, \tag{1}$$

where $p_\mu(\sigma)=i\delta/\delta x^\mu(\sigma)$. In the standard covariant formalism, however, one supplements constraint (1), representing σ-reparametrization invariance, with the dynamical constraint

$$\{p^2+x'^2/(2\pi\alpha')^2\}\Phi[x^\mu(\sigma)]=0, \tag{2}$$

implementing τ-reparametrization invariance on world sheets. These two constraints are equivalent to the conditions $L_n\Phi=0$ for every generator L_n of the Virasoro algebra. The difficulty introduced by constraint (2) is that, at the quantum level, these conditions are incompatible with the existence of a central charge in the Virasoro algebra. We can salvage this situation by requiring that only L_n with $n>0$ may annihilate physical states. However, since (1) is equivalent to the conditions $(L_n-L_{-n})\Phi=0$, we immediately recognize that σ-reparametrization invariance is not maintained in the physical sector so defined. Thus in the standard covariant formalism one necessarily abandons the very notion of line functionals, thereby obscuring the geometrical principles underlying these string field theories.

In view of these and other problems, it is worthwhile to consider alternative formulations of string field dynamics which are geometric in the sense of condition (1). To do so, as we have argued, we must somehow eliminate the need for dynamical constraints such as Eq. (2). In this Letter, we explore one such alternative: a multicomponent line functional Ψ_α[line] which obeys a generalized Dirac equation. Our development follows closely the approach of Marshall and Ramond,[6] in which manifest σ-reparametrization (RP) invariance is maintained at every stage. Dynamical constraints are avoided by requiring that the generalized Dirac equation we shall present reproduces Nambu-Goto string dynamics in the $x^0=\tau$ gauge. Other studies of line functionals may be found elsewhere.[7,8]

Our starting point is the Nambu-Goto action evaluated in the "laboratory gauge," $x^0(\tau,\sigma)=\tau$:

$$S_{\text{lab}}=-(2\pi\alpha')^{-1}\int_{\tau_i}^{\tau_f}d\tau\int_0^1 d\sigma[(\dot{\mathbf{x}}\cdot\mathbf{x}')^2+|\mathbf{x}'|^2(1-|\dot{\mathbf{x}}|^2)]^{1/2}, \tag{3}$$

where $\dot{x}^i=\partial x^i/\partial\tau$ and $x^{i\prime}=\partial x^i/\partial\sigma$, $i=1,\ldots,d-1$. In this gauge, the canonical conjugate momenta $\mathbf{p}(\sigma)$ satisfy one, and only one, constraint,

$$\mathbf{x}'(\sigma)\cdot\mathbf{p}(\sigma)=0, \tag{4}$$

corresponding to the residual σ-RP invariance of the action (3). The Hamiltonian is given by

$$H=\int_0^1 d\sigma(\mathbf{x}'^2)^{1/2}\{\mathbf{p}^2/\mathbf{x}'^2+(2\pi\alpha')^{-2}\}^{1/2}. \tag{5}$$

Note that the Hamiltonian is RP invariant, as may be verified by commutation of the constraint (4) with H.

We may not attempt to infer a wave equation for string fields $\Phi[\tau,\mathbf{x}(\sigma)]$ in this gauge by replacing $\mathbf{p}(\sigma)$ by $-i\delta/\delta\mathbf{x}(\sigma)$ in Eqs. (4) and (5), which are now viewed as operators acting on Φ:

$$\mathbf{x}'\cdot\mathbf{p}\Phi=0, \tag{6}$$

$$i\,\partial\Phi/\partial\tau=H\Phi. \tag{7}$$

Equation (6) simply states that Φ is a line functional of spacelike curves $\mathbf{x}(\sigma)$. Equation (7), however, is ill-defined since we have the problem of interpreting the square root in H. Following Dirac's treatment of the relativistic point particle, we replace H by $\int_0^1 d\sigma(\mathbf{x}'^2)^{1/2}M(\sigma)$, where

$$M(\sigma)=\boldsymbol{\alpha}(\sigma)\cdot\mathbf{p}/(\mathbf{x}'^2)^{1/2}+\beta(\sigma)/2\pi\alpha', \tag{8}$$

is a linear matrix operator which obeys the relation

$$M(\sigma)^2=\mathbf{p}^2/\mathbf{x}'^2+(2\pi\alpha')^{-2} \tag{9}$$

up to terms proportional to $\mathbf{x}'\cdot\mathbf{p}$. Thus we arrive at the wave equation,

$$i\,\partial\Psi_{\text{lab}}/\partial\tau=\int_0^1 d\sigma(\mathbf{x}'^2)^{1/2}M(\sigma)\Psi_{\text{lab}}, \tag{10}$$

for a multicomponent line functional $\Psi_{\text{lab}}[\tau;\mathbf{x}(\sigma)]$ in the laboratory gauge.

The correspondence condition (9) represents a set of nontrivial constraints on the matrices $\alpha^i(\sigma)$ and $\beta(\sigma)$, but we expect as in the point-particle case that Lorentz covariance will more stringently limit their possible forms. We consider therefore the manifestly covariant equation

$$\int_0^1 d\sigma\left\{\Gamma^\mu(\sigma)p_\mu(\sigma)-\frac{[-x'^2(\sigma)]^{1/2}}{2\pi\alpha'}\Lambda(\sigma)\right\}\Psi=0, \tag{11}$$

where Ψ obeys the covariant analog of Eq. (6):

$$x'(\sigma)\cdot p(\sigma)\Psi=0. \tag{12}$$

To restrict the possible forms of the matrices $\Gamma^\mu(\sigma)$ and $\Lambda(\sigma)$, we shall require that (i) Eq. (11) be Lorentz covariant, translation invariant, and RP invariant; (ii) $\Gamma^\mu(\sigma)$ and $\Lambda(\sigma)$ commute with $p_\mu(\sigma)$; (iii) $\Gamma^\mu(\sigma)$ and $\Lambda(\sigma)$ depend on σ only through $x'_\mu(\sigma)$ and its derivatives; and (iv) Eq. (11) reproduces (10) in the laboratory gauge with a local matrix $M(\sigma)$ satisfying Eq. (9). Conditions (ii) and (iii) are to some extent arbitrary and may be relaxed in a more general theory. Condition (iii), for example, forbids the introduction of spin-density degrees of freedom as encountered in versions of the spinning string.[9]

Conditions (i) and (iii) imply that $\Gamma^\mu(\sigma)$ and $\Lambda(\sigma)$ depend on σ only through the RP-invariant tangent vector, $t_\mu=x'_\mu/(-x'^2)^{1/2}$, and its derivatives with respect to arc length. Among them only $t^\mu(\sigma)$ itself satisfies condition (ii); thus $\Gamma^\mu(\sigma)$ and $\Lambda(\sigma)$ may be expanded in a Taylor series in t^μ with constant matrix coefficients. Condition (iv) then fixes the form of $\Gamma^\mu(\sigma)$ up to three independent terms, whereas $\Lambda(\sigma)$ may contain terms of arbitrary order in $t^\mu(\sigma)$. We take the simplest case $\Lambda(\sigma)=1$ to obtain the following wave equation for closed strings:

$$\{a\Gamma^\mu P_\mu+\tfrac{1}{2}b\Gamma^{\mu\nu}_{(-)}M^{(-)}_{\mu\nu}+\tfrac{1}{2}ic\Gamma^{\mu\nu}_{(+)}M^{(+)}_{\mu\nu}-l/2\pi\alpha'\}\Psi=0. \tag{13}$$

Here

$$P_\mu=\int_0^1 d\sigma\,p_\mu(\sigma),\quad M^{(\pm)}_{\mu\nu}=\int_0^1 d\sigma(t_\mu p_\nu\pm t_\nu p_\mu),\quad l=\int_0^1 d\sigma(-x'^2)^{1/2},$$

while a, b, and c are unspecified real constants. The constant matrices $\Gamma^{\mu\nu}_{(+)}$ ($\Gamma^{\mu\nu}_{(-)}$) are symmetric (antisymmetric) in their Lorentz indices. Without loss of generality, we may also require $\eta_{\mu\nu}\Gamma^{\mu\nu}_{(+)}=0$ since $t\cdot p$ always annhilates Ψ.

One subtlety encountered in implementing condition (iv) is the proper interpretation of $p_0(\sigma)$ in the laboratory gauge; the naive substitution $p_0(\sigma)\to i\delta/\delta\tau$ is incorrect as it violates RP invariance. This difficulty is resolved by the introduction of string coordinates $y^\mu(s)$, $0\leq s\leq 1$, which are parametrized by arc length $ls(\sigma)$: $y^\mu[s(\sigma)]=x^\mu(\sigma)$. With the identification

$$\Psi_{\text{lab}}[\tau;\mathbf{x}(\sigma)]=\Psi[y^\mu(s)]\big|_{y^0(s)=\tau}, \tag{14}$$

one may easily show that

$$\frac{1}{[-x'(\sigma)^2]^{1/2}}\frac{\delta}{\delta x^0(\sigma)}\Psi\to\frac{1}{l}\frac{\delta}{\delta y^0(s)}\Psi\bigg|_{y^0(s)=\tau}=\frac{1}{l}\frac{\partial}{\partial\tau}\Psi_{\text{lab}}. \tag{15}$$

Notice that (14) and (15) are justified only for line functionals.

With the substitution (15) made, we find that for closed strings $M^{(\pm)}_{k0}\Psi$ vanishes in the laboratory gauge. Consequently, Eq. (13) may be cast in the form (10) up to an overall multiplicative constant $1/a$ which is absorbed by rescaling of τ. The matrix $M(\sigma)$ so determined fulfills the correspondence condition (9) provided that (i) the Γ

matrices satisfy the anticommutation relations

$$\{\Gamma^{\mu},\Gamma^{\nu}\}=2\eta^{\mu\nu},\quad \{\Gamma^{\mu\nu}_{(-)},\Gamma^{\lambda\sigma}_{(-)}\}=2(\eta^{\mu\lambda}\eta^{\nu\sigma}-\eta^{\mu\sigma}\eta^{\nu\lambda}),$$
$$\{\Gamma^{\mu\nu}_{(+)},\Gamma^{\lambda\sigma}_{(+)}\}=2[\eta^{\mu\lambda}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\lambda}-(2/d)\eta^{\mu\nu}\eta^{\lambda\sigma}] \tag{16}$$

(all other anticommutators vanish), and (ii) the parameters a, b, and c obey the relation

$$a^2+b^2+c^2=1. \tag{17}$$

One can show that the algebras for $\Gamma^{\mu\nu}_{(-)}$ and $\Gamma^{\mu\nu}_{(+)}$ are equivalent to Clifford algebras in $d(d-1)/2$ and $(d-1)(d+2)/2$ dimensions. Finally, the generators of Lorentz transformations are given by

$$J^{\mu\nu}=\int_0^1 d\sigma(x^{\mu}p^{\nu}-x^{\nu}p^{\mu})+\tfrac{1}{4}i[\Gamma^{\mu},\Gamma^{\nu}]+\tfrac{1}{4}i[\Gamma^{\mu\alpha}_{(-)},\Gamma^{\nu}_{(-)\alpha}]+\tfrac{1}{4}i[\Gamma^{\mu\alpha}_{(+)},\Gamma^{\nu}_{(+)\alpha}]. \tag{18}$$

In passing, the algebra of $\Gamma^{\mu\nu}_{(-)}$ in (16) has been introduced in the context of string field equations in work by Hosotani.[10] Our Eq. (13) differs from the equations there which were derived from a correspondence to the Hamilton-Jacobi equations for classical strings. Equation (13) also differs from the equations of Marshall and Ramond[6] and the second-order equation $2(L_0-1)P\Phi=0$ encountered in the standard covariant formalism.[1,2]

In three dimensions Eq. (13) can be solved for $c=0$ in the laboratory gauge, $x^0(\sigma)=y^0$. For Γ^{μ} and $\Gamma^{\mu\nu}_{(-)}$ we choose the following realization of the algebra (16): $\Gamma^{\mu}=\gamma^{\mu}\otimes I$, $\Gamma^{0k}_{(-)}=i\gamma_5\otimes\tau_k$, and $\Gamma^{12}_{(-)}=\gamma_5\otimes\tau_3$, where γ^{μ} are the 4×4 Dirac matrices in the spinor representation and τ_k the 2×2 Pauli matrices. We look for solutions to Eq. (13) of the form

$$\Psi_{\text{lab}}=e^{-ip\cdot y}\begin{bmatrix}\xi(l)\\ \eta(l)\end{bmatrix}, \tag{19}$$

where $\xi(l)$ and $\eta(l)$ carry two spinor indices on which $\boldsymbol{\sigma}$ (in $\boldsymbol{\gamma}$) and $\boldsymbol{\tau}$ act. Here l is the string length and $\mathbf{y}$ is the center-of-mass coordinate; in the arc-length parametrization, $\mathbf{y}=\int_0^1 d\sigma\,\mathbf{x}(\sigma)$. Periodicity of $\mathbf{x}'(\sigma)$ also implies that

$$q=(1/2\pi)\int_0^1 d\sigma\, t_i\epsilon_{ij}t_j' \tag{20}$$

is an integer, known as the rotation index of the closed plane curve $\mathbf{x}(\sigma)$. In particular, $q=\pm 1$ for simple (i.e., nonintersecting) curves.

For smooth curves with $q\neq 0$, Eq. (13) is reduced to

$$a(p_0-\mathbf{p}\cdot\boldsymbol{\sigma})\eta-\left\{2\pi qb\tau_3\frac{d}{dl}+\frac{1}{2\pi\alpha'}l\right\}\xi=0,$$
$$a(p_0+\mathbf{p}\cdot\boldsymbol{\sigma})\xi+\left\{2\pi qb\tau_3\frac{d}{dl}-\frac{1}{2\pi\alpha'}l\right\}\eta=0. \tag{21}$$

If we employ the boundary conditions $\xi,\eta\to 0$ as $l\to\infty$, the eigenvalues of $p^2=m^2$ corresponding to the *Ansatz* (19) are found to be quantized:

$$m^2=(2|bq|/\alpha' a^2)n,\quad n=0,1,2,\ \ldots. \tag{22}$$

Besides obtaining the equally spaced mass-squared spectrum characteristic of free string theories, we see that Eq. (13) with $c=0$ also admits massless particle states.

With our choice of the representation of Γ matrices, the rotation generator (18) for the functional (19) becomes

$$J_{12}=-i\left\{y_1\frac{\partial}{\partial y_2}-y_2\frac{\partial}{\partial y_1}\right\}+\frac{1}{2}\begin{bmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{bmatrix}\otimes I+\tfrac{1}{2}I\otimes\tau_3, \tag{23}$$

whose spin piece is a direct product of two spin-$\frac{1}{2}$ representations. We see that eigenstates (22) have spin $J=0$ and 1. Since eigenvalues of orbital angular momenta are always integral, we conclude that in $1+2$ dimensions Eq. (13) with $c=0$ defines a bosonic string theory. This statement, however, depends on the dimensionality d and on the choice of constants a, b, and c; in $1+3$ dimensions with $c=0$, for instance, Ψ describes a fermionic string.

One might wonder if Eq. (13) can be derived from the variation of an action. To discuss actions for line functionals, we must first supply a precise and unique definition of a measure in the space of curves. This has been accomplished and will be presented elsewhere.[11]

We have attempted to define a string field theory as a theory of line functionals, arriving at the manifestly RP-invariant and Lorentz-covariant field equations (11) and (13). It remains for further work to ascertain whether the approach to string field dynamics that we have outlined above coincides with the more conventional treatments of the covariant string. Of particular interest are the quantization of the model, the existence of tachyonic states, the incorporation of gauge symmetries, and the determination of critical dimensions. These and other topics are currently under investigation.

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